EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH2581-WE01

Title: Algebra II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 (a) Which of the following rings are integral domains? Justify your answers.

(i) $\mathbb{Z}/6$,

(ii) $\mathbb{Q}[x]/(x^5 + px + p)$ for p a prime,

(iii) $(\mathbb{Z}/3)[x]/(x^2 + \bar{2})$,

(iv) $\mathbb{Z}/7 \times \mathbb{Z}/3$,

(v) $\mathbb{Z}[i]$.

(b) Find all units of the following rings. There is no need to justify your answer.

(i) $\mathbb{Z}/7$,

(ii) $\mathbb{Q}[x]$,

(iii) $\mathbb{Z}/3 \times \mathbb{Z}/4$,

(iv) $\mathrm{M}_2(\mathbb{Q})$,

(v) $\mathbb{Z}[x]$.

Q2 Let R and S be rings, and let $\phi : R \to S$ be a ring homomorphism.

(a) Show that if J is an ideal of S, then $\phi^{-1}(J) = \{r \in R \mid \phi(r) \in J\}$ is an ideal of R.

(b) Assume now that ϕ is surjective. Show that, if I is an ideal of R, then $\phi(I) = \{\phi(r) \mid r \in I\}$ is an ideal of S.

Q3 (a) Let $G = S_7$ and $\gamma = (1\,2\,3)(4\,2\,3)(5\,4)(6\,7)$.

(i) What is the order of γ in G?

(ii) Give the size of the conjugacy class of γ in G. [Justify your answer.]

(b) Determine the rank and the torsion coefficients of the kernel of the map

$$\begin{aligned} f \quad : \quad \mathbb{Z}^3 \quad &\to \quad \mathbb{Z}\,, \\ (a, b, c) &\mapsto 12a + 15b + 6c. \end{aligned}$$

Q4 (a) Let G be an abelian group. Show that the set of elements of order 1 or 2 form a subgroup of G.

(b) Assume all elements of a group G have order at most 2. Show that G is abelian.

SECTION B

Q5 (a) (i) Factorise the following polynomials into irreducible factors in $(\mathbb{Z}/2)[x]$.

$$f(x) = x^5 + x^4 + x^3 + x^2 + x + \bar{1}, \text{ and } g(x) = x^6 + x^4 + x + \bar{1}.$$

(ii) With notation as above, let $I = (f(x), g(x))$ be the ideal in $(\mathbb{Z}/2)[x]$ generated by $f(x)$ and $g(x)$. Consider the quotient ring $R = (\mathbb{Z}/2)[x]/I$. Give a set of representatives for the elements in R. Is R a field? Justify your answers.

(b) Let $R = \mathbb{Z}[\sqrt{-5}]$, and let $I = (3, 2 + \sqrt{-5})$. Show that the ideal I is not a principal ideal.

Q6 (a) Let $n > 1$ be an integer and consider the ideal $I = (n, x)$ in $\mathbb{Z}[x]$. Give conditions on n such that I is a maximal ideal. Justify your answer.

(b) Show that the quotient ring $\mathbb{Q}[x]/(x^3 - 3x^2 - 3x + 9)$ is isomorphic to the ring $\mathbb{Q} \times \mathbb{Q}[\sqrt{3}]$, where $\mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$.

Q7 Let $G = (\mathbb{Z}/15)^\times$, the group of units in the ring $\mathbb{Z}/15$.

(a) Write down the elements of G, say as classes in $\mathbb{Z}/15$, giving both the order and the inverse of each element.

(b) Write G as a product of cyclic groups. [Justify your result.]

(c) State Cayley's Theorem.

(d) Use the procedure in the proof of Cayley's Theorem to find generators of a subgroup of S_8 to which G is isomorphic.

Q8 Let G be a group of order 20.

(a) Show that G contains a group H of order 5, stating carefully any results that you use.

(b) For any $g \in G$ show that its conjugate gHg^{-1} is also a subgroup of G.

(c) Show that the intersection of two different conjugates of H contains only one element.

(d) Given the previous part, find a sharp bound for how many different conjugates of H can be contained in G.

(e) Show that the number of conjugates of H divides $|G|$. (Hint: You may want to consider a suitable group action.)

(f) Using the previous parts, show that every group with 20 elements contains a *normal* subgroup of order 5.