Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH2617-WE01

Title:

Elementary Number Theory II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 (a) Calculate $\phi(2320)$, where ϕ is the Euler phi function. Show your work.

(b) Find the last two digits of $7^{(3^4)}$. Show your work.

(c) Determine, with justification, if each of the following is a sum of two squares. If so, find a representation for it in the form $n = a^2 + b^2$:

(i) 265, (ii) 238.

Q2 (a) Find all primitive Pythagorean triples of the form $(56, y, z)$. Show your work.

(b) Determine whether the congruence $x^2 + 18x + 51 \equiv 0 \pmod{101}$ is solvable in $x \in \mathbb{Z}$. Show your work.

(c) Fix $b \in \mathbb{N}$ and let n be an integer representable as $n = x^2 - by^2$, for some $x, y \in \mathbb{Z}$. Show that, for every prime $p|n$, at least one of (i) $p|x$ and (ii) $\left(\frac{b}{p}\right) = +1$ must hold.

SECTION B

Q3 Let $a, b \in \mathbb{N}$ be distinct positive integers with $5 \nmid (b - a)$.

(a) Show that for every positive integer n, $\gcd(a + 5n, b + 5n)$ divides $b - a$.

(b) Show that there are only finitely many prime numbers p such that p can divide *both* $a + 5n$ and $b + 5n$ for some $n \in \mathbb{N}$.

(c) Let S be the finite set of primes from part (b). Show that for each $p \in S$ there exists a positive integer n such that p divides *at most one* of $a+5n$ and $b+5n$.

(d) Show that there is a positive integer n such that $a+5n$ and $b+5n$ are coprime. (*Hint:* Chinese remainder theorem.)

(e) Upgrade (d) by showing that there are *infinitely many* n such that $a + 5n$ and $b + 5n$ are coprime.
(*Hint:* Show that, as a function of n, $\gcd(a + 5n, b + 5n)$ is periodic modulo $|b - a|$.)

Q4 Given a positive integer n, let $\sigma(n)$ denote the sum of all of the positive integer divisors of n. Notationally, we write

$$\sigma(n) = \sum_{d|n} d.$$

For example, $\sigma(14) = 1 + 2 + 7 + 14 = 24$.

(a) Compute $\sigma(49)$ and $\sigma(12)$.

(b) Let p be a prime number and let $k \geq 1$. Establish a formula for $\sigma(p^k)$.

(c) Show that $\sigma(2^k)$ is odd for all $k \geq 1$. For primes $p > 2$, for which $k \geq 1$ is it true that $\sigma(p^k)$ is odd? Prove your answer.

(d) Show that if $m, n \in \mathbb{N}$ are coprime then every d dividing mn may be written as $d = d_1 d_2$, where $d_1 | m$ and $d_2 | n$.

(e) Use (d) to show that if m and n are coprime then $\sigma(mn) = \sigma(m)\sigma(n)$.

(f) Find all n such that $\sigma(n)$ is odd.