

EXAMINATION PAPER

Examination Session: May/June

2025

Year:

Exam Code:

MATH2617-WE01

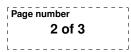
Title:

Elementary Number Theory II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions.		
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.		
	Write your answer in the white-covered answer booklet with barcodes.		
	Begin your answer to each question on a new page.		

Revision:

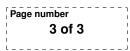


SECTION A

- Q1 (a) Calculate $\phi(2320)$, where ϕ is the Euler phi function. Show your work.
 - (b) Find the last two digits of $7^{(3^4)}$. Show your work.
 - (c) Determine, with justification, if each of the following is a sum of two squares. If so, find a representation for it in the form $n = a^2 + b^2$:

(i) 265, (ii) 238.

- Q2 (a) Find all primitive Pythagorean triples of the form (56, y, z). Show your work.
 - (b) Determine whether the congruence $x^2 + 18x + 51 \equiv 0 \pmod{101}$ is solvable in $x \in \mathbb{Z}$. Show your work.
 - (c) Fix $b \in \mathbb{N}$ and let *n* be an integer representable as $n = x^2 by^2$, for some $x, y \in \mathbb{Z}$. Show that, for every prime p|n, at least one of (i) p|x and (ii) $\left(\frac{b}{p}\right) = +1$ must hold.



SECTION B

- **Q3** Let $a, b \in \mathbb{N}$ be distinct positive integers with $5 \nmid (b a)$.
 - (a) Show that for every positive integer n, gcd(a + 5n, b + 5n) divides b a.
 - (b) Show that there are only finitely many prime numbers p such that p can divide both a + 5n and b + 5n for some $n \in \mathbb{N}$.
 - (c) Let S be the finite set of primes from part (b). Show that for each $p \in S$ there exists a positive integer n such that p divides at most one of a + 5n and b + 5n.
 - (d) Show that there is a positive integer n such that a + 5n and b + 5n are coprime. (*Hint:* Chinese remainder theorem.)
 - (e) Upgrade (d) by showing that there are *infinitely many n* such that a + 5n and b + 5n are coprime.
 (*Hint:* Show that, as a function of n, gcd(a + 5n, b + 5n) is periodic modulo |b a|.)
- **Q4** Given a positive integer n, let $\sigma(n)$ denote the sum of all of the positive integer divisors of n. Notationally, we write

$$\sigma(n) = \sum_{d|n} d$$

For example, $\sigma(14) = 1 + 2 + 7 + 14 = 24$.

- (a) Compute $\sigma(49)$ and $\sigma(12)$.
- (b) Let p be a prime number and let $k \ge 1$. Establish a formula for $\sigma(p^k)$.
- (c) Show that $\sigma(2^k)$ is odd for all $k \ge 1$. For primes p > 2, for which $k \ge 1$ is it true that $\sigma(p^k)$ is odd? Prove your answer.
- (d) Show that if $m, n \in \mathbb{N}$ are coprime then every d dividing mn may be written as $d = d_1 d_2$, where $d_1 | m$ and $d_2 | n$.
- (e) Use (d) to show that if m and n are coprime then $\sigma(mn) = \sigma(m)\sigma(n)$.
- (f) Find all n such that $\sigma(n)$ is odd.