Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH2647-WE01

Title: Probability II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(a) What are the defining properties of the σ-algebra $\mathcal{F}$?

(b) What is the probability generating function of a random variable

$$X \colon \Omega \to \{0, 1, 2, 3, \ldots\}?$$

Suppose X and Y are independent random variables, which both take values in $\{0, 1, 2, 3, \ldots\}$. Express the probability generating function of $X + Y$ in terms of the individual probability generating functions of X and Y.

(c) What is the cumulative distribution function F of a random variable X? State the main properties of F (you need not prove them).

Q2 Let $(\Omega, \mathcal{F}, P)$ be a probability space. Carefully state the following results.

(a) The first Borel–Cantelli Lemma for a sequence of events $A_n \in \mathcal{F}$, $n \geq 1$.

(b) The Bounded Convergence Theorem for a sequence of random variables X_n, $n \geq 1$.

(c) The Weak Law of Large Numbers for a sequence of random variables X_n, $n \geq 1$.

SECTION B

Q3 Let $(\Omega, \mathcal{F}, P)$ be a probability space. In the following questions, carefully justify your answers using appropriate definitions and results from the lectures. You may use any results from the lectures without proof, as long as they are stated clearly.

(a) What does it mean for a sequence of random variables X_n, $n \geq 1$, to converge in probability?

Now let X_n, $n \geq 1$, be a sequence of random variables such that

$$P(X_n = 0) = 1 - \frac{1}{n}$$

for every $n \geq 1$. Does this sequence converge in probability?

(b) What does it mean for the sequence of random variables X_n, $n \geq 1$, to converge almost surely?

Now let X_n, $n \geq 1$, be a sequence of random variables such that

$$\mathbb{E}[X_n^2] \leq \frac{1}{n^2}$$

for every $n \geq 1$. Does this sequence converge almost surely?

Q4 Let $(\Omega, \mathcal{F}, P)$ be a probability space. In the following questions, carefully justify your answers using appropriate definitions and results from the lectures. You may use any results from the lectures without proof, as long as they are stated clearly.

(a) Let $X\colon \Omega \to \mathbb{R}$ be a random variable such that $\mathbb{E}[|X|] < \infty$. Let A_n for $n \geq 1$ be a sequence of pairwise disjoint events, that is, $A_n \cap A_m = \emptyset$ if $n \neq m$. Let $A = \cup_{n=1}^{\infty} A_n$. Recall that for an event E, its indicator is the random variable $\mathbf{1}_E$, defined such that

$$\mathbf{1}_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{if } \omega \notin E. \end{cases}$$

Show that, for every $m \geq 1$,

$$\sum_{n=1}^{m} \mathbb{E}[X\mathbf{1}_{A_n}] = \mathbb{E}[X\mathbf{1}_{\cup_{n=1}^{m} A_n}],$$

and then show that

$$\sum_{n=1}^{\infty} \mathbb{E}[X\mathbf{1}_{A_n}] = \mathbb{E}[X\mathbf{1}_A].$$

(b) Let $X_n\colon \Omega \to \mathbb{R}$ for $n \geq 1$ be independent random variables.

(i) Suppose that there is some $a \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} P(X_n > a) < \infty$. Show that

$$\sup_{n \geq 1} X_n < \infty$$

almost surely.

(ii) Suppose that $\sup_{n \geq 1} X_n < \infty$ almost surely. Is it necessarily true that

$$\sum_{n=1}^{\infty} P(X_n > a) < \infty$$

for some $a \in \mathbb{R}$?