Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH2707-WE01

Title:
Markov Chains II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 Suppose P is the stochastic matrix

$$P = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}.$$

(a) Determine the communicating classes of P.

(b) Determine which states are recurrent, and which are transient.

(c) Determine the period of each state.

For each problem you should show all of your workings and justify your calculations with suitable explanations.

Q2 Consider the transition matrix P on $I = \{1, 2, 3\}$ given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Suppose $(X_n)_{n\geq 0}$ is a Markov chain with transition matrix P. Let $f \colon I \to \mathbb{R}$ be given by $f(1) = f(3) = -1$ and $f(2) = 1$. Compute the quantity

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(X_n),$$

and justify your computation by citing an appropriate theorem.

SECTION B

Q3 You have been asked to program a robotic security guard for an art gallery. The art gallery is rectangular in shape, and is tiled with square floor tiles. See Figure 1 for an illustration of the case when the gallery is five tiles wide. This problem concerns galleries of width $2n+1$ for some $n \geq 1$.

The owners of the gallery have asked that the robot spends, on average, **twice as much time** on a tile at distance k from the central tile as at a tile at distance $k+1$ for all $k = 0, 1, \ldots, n-1$. The robot must behave as a Markov Chain, and its design restricts it to moving at most one tile horizontally in any unit of time.

(a) Suppose $n \geq 1$. Write down a formula for the entries of a transition matrix that meets the requirements given in the case of a gallery of width $2n+1$.

You should index your matrix by $I = \{-n, -n+1, \ldots, n-1, n\}$, with $-n$ the leftmost tile and n the rightmost tile. You must carefully justify why your solution leads to the requirements being met.

(b) Let $p_{ij}(n)$ be the ij^{th} entry of a matrix that provides a solution to part (a), and P be the matrix with entries $p_{ij} = \lim_{n\to\infty} p_{ij}(n)$. Show that P is a stochastic matrix.

(c) Let P be a transition matrix that provides a solution to part (b). Suppose $(X_m)_{m\geq 0}$ is a Markov Chain with transition matrix P. Is this Markov Chain recurrent or not?

For each problem you should show all of your workings and justify your calculations with suitable explanations.

2	1	0	1	2

Figure 1: The case of a gallery of width 5. Each tile is labelled with its distance to the central tile.

Q4 For this problem, the state space is $I = \{0, 1, 2, 3, 4\}$, and P is the stochastic matrix with entries p_{ij} described by the following figure.

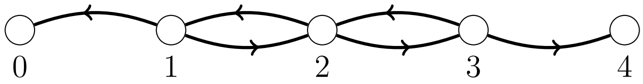

Figure 2: Illustration of P for **Q4**. All arrows have the value $\frac{1}{2}$.

Recall that H^A is the hitting time of a set of states $A \subset I$.

(a) Compute $\mathbb{E}_1[H^{\{0,4\}}]$.

(b) Compute $\mathbb{P}_i[H^{\{4\}} < H^{\{0\}}]$ for each $i \in I$. **Hint**: explain why you can apply a standard theorem to compute this.

(c) Define $\tilde{p}_{ii} = 1$ for $i \in \{0, 4\}$, $\tilde{p}_{ij} = 0$ if $i \in \{0, 4\}$ and $j \neq i$, and otherwise define

$$\tilde{p}_{ij} = \frac{p_{ij}\mathbb{P}_j[H^{\{4\}} < H^{\{0\}}]}{\mathbb{P}_i[H^{\{4\}} < H^{\{0\}}]}.$$

Is the matrix $\tilde{P}$ with entries $\tilde{p}_{ij}$ a stochastic matrix?

(d) Compute $\mathbb{E}_i[H^{\{4\}} \mid H^{\{4\}} < H^{\{0\}}]$.

For each problem you should show all of your workings and justify your calculations with suitable explanations.