Statistical Inference II: Formula Sheet

Standard Distributions:

Exponential For $x \geq 0$ and $\lambda > 0$, $X \sim \mathcal{Exp}(\lambda)$ has pdf

$$f(x) = \lambda e^{-\lambda x},$$

with $\mathbb{E}[X] = \frac{1}{\lambda}$ and $\mathbb{V}\mathrm{ar}[X] = \frac{1}{\lambda^2}$.

Gamma distribution: For $y > 0$, $\alpha > 0$ and $\beta > 0$, $Y \sim \mathcal{Gamma}(\alpha, \beta)$ has pdf:

$$f(y \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y},$$

where $\Gamma(\alpha)$ is the Gamma function, and $\mathbb{E}[Y] = \frac{\alpha}{\beta}$ and $\mathbb{V}\mathrm{ar}[Y] = \frac{\alpha}{\beta^2}$.

Inverse-Gamma distribution: For $y > 0$, $\alpha > 0$ and $\beta > 0$, $Y \sim \mathcal{InvGamma}(\alpha, \beta)$ has pdf:

$$f(y \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} e^{-\beta/y},$$

where $\Gamma(\alpha)$ is the Gamma function, and $\mathbb{E}[Y] = \frac{\beta}{\alpha-1}$ for $\alpha > 1$ and $\mathbb{V}\mathrm{ar}[Y] = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$ for $\alpha > 2$.

Gamma function: $\Gamma(x)$ is the Gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t$, with properties:

- $\Gamma(x+1) = x\Gamma(x)$
- $\Gamma(1) = 1$; $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
- For integer z, $\Gamma(z) = (z-1)!$

Beta distribution: For $z \in [0,1]$, $a > 0$ and $b > 0$, $Z \sim \mathcal{Beta}(a, b)$ has pdf:

$$f(z \mid a, b) = \frac{1}{B(a,b)} z^{a-1}(1-z)^{b-1},$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, and $\mathbb{E}[Z] = \frac{a}{a+b}$ and $\mathbb{V}\mathrm{ar}[Z] = \frac{ab}{(a+b)^2(a+b+1)}$.

Normal distribution: For $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ and $\sigma > 0$, $X \sim \mathcal{N}(\mu, \sigma^2)$ has pdf:

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\},$$

and $\mathbb{E}[X] = \mu$ and $\mathbb{V}\mathrm{ar}[X] = \sigma^2$.

Chi-squared For $x > 0$ and $k \in \mathbb{N}$, $X \sim \chi^2_k$ has pdf

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2},$$

with $\mathbb{E}[X] = k$ and $\mathbb{V}\mathrm{ar}[X] = 2k$.

Multivariate Normal distribution: For $\mathbf{x} \in \mathbb{R}^p$, $\boldsymbol{\mu} \in \mathbb{R}^p$, and positive definite $p \times p$ variance matrix $\boldsymbol{\Sigma}$, $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ has pdf:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{p}{2}} (\det(\boldsymbol{\Sigma}))^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\},$$

with $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$ and $\mathbb{V}\mathrm{ar}[\mathbf{X}] = \boldsymbol{\Sigma}$.
For $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with $\mathbf{Y} = A\mathbf{X} + \mathbf{b}$ where A is a $r \times p$ real matrix and $\mathbf{b} \in \mathbb{R}^r$:

$$\mathbf{Y} \sim \mathcal{N}_r\left(A\boldsymbol{\mu} + \mathbf{b}, B\boldsymbol{\Sigma}B^T\right)$$

Distribution Theory

Change of variables: For r.v. $X \in \mathcal{X}$ and $Y = y(X) \in \mathcal{Y}$, under appropriate conditions on $y(\cdot)$:

$$f_Y(y) = f_X(x(y)) \left|\frac{\mathrm{d}x(y)}{\mathrm{d}y}\right|.$$

Multivariate: For $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^p$ and $\mathbf{Y} = \mathbf{y}(\mathbf{X}) \in \mathcal{Y} \subseteq \mathbb{R}^p$, then under appropriate conditions on $\mathbf{y}(\cdot)$:

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(x(\mathbf{y})) \left|\det(\mathbf{J})\right|,$$

where $\mathbf{J}$ is the $p \times p$ Jacobian matrix, $[\mathbf{J}]_{ij} = \dfrac{\partial x_i}{\partial y_j}$.

Normal Sampling

For $X_1, \ldots, X_n$ i.i.d $\mathcal{N}(\mu, \sigma^2)$:

1. $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$,
2. $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$,
3. $T = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim \mathrm{t}_{n-1}$.

Likelihood

Information:

Observed information:

$$I\left(\hat{\theta}\right) = -\frac{\partial^2}{\partial\theta^2} \log \ell(\theta)\Big|_{\theta=\hat{\theta}} = -\mathcal{L}''\left(\hat{\theta}\right)$$

Expected (Fisher) information:

$$\mathcal{I}_n(\theta) = \mathbb{E}_{\mathbf{X}}[I(\theta)].$$

:

Cramér-Rao Lower Bound: Unbiased estimator T of parameter θ with regular likelihood from sample of size n has:

$$\mathbb{V}\text{ar}\left[T\right] \geq \frac{1}{\mathcal{I}_n\left(\theta\right)}.$$

Large-sample behaviour: For i.i.d sample $\mathbf{X} = (x_1, \ldots, x_n)$ from $f(x \mid \theta)$, then under appropriate conditions MLE $\hat{\theta}$ for scalar θ has a limiting distribution as $n \to \infty$:

$$\hat{\theta} \rightsquigarrow \mathcal{N}\left(\theta, \frac{1}{\mathcal{I}_n\left(\theta\right)}\right).$$

Delta method Under appropriate conditions, a collection of continuously differentiable transformations $\boldsymbol{\phi} = \mathbf{g}(\boldsymbol{\theta})$ of parameter $\boldsymbol{\theta}$, $g : \mathbb{R}^p \to \mathbb{R}^q$, has pdf for $p = 1$

$$\hat{\phi} = g(\hat{\theta}) \sim \mathcal{N}\left(g(\theta),\ \frac{1}{\mathcal{I}_n\left(\theta\right)}\left(\frac{\mathrm{d}g(\hat{\theta})}{\mathrm{d}\theta}\right)^2\right),$$

and for $p > 1$

$$\hat{\boldsymbol{\phi}} = \mathbf{g}(\hat{\boldsymbol{\theta}}) \sim \mathcal{N}_q\left(\mathbf{g}(\boldsymbol{\theta}),\ \mathbf{J}(\hat{\boldsymbol{\theta}})\mathcal{I}_n\left(\theta\right)^{-1}\mathbf{J}(\hat{\boldsymbol{\theta}})^T\right),$$

where $\mathbf{J}(\hat{\boldsymbol{\theta}}) = \left[\dfrac{\partial g_i(\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}_j}\right]_{ij}$.

Bayesian statistics

Beta-Binomial: For $X \mid \pi \sim \mathcal{Bin}\left(n, \pi\right)$ and prior $\pi \sim \mathcal{Beta}\left(a, b\right)$:

$$\pi \mid X \sim \mathcal{Beta}\left(a + x, b + n - x\right).$$

Gamma-Poisson For c.i.i.d $X_1, \ldots, X_n$, with each $X \mid \lambda \sim \mathcal{Po}\left(\lambda\right)$ and prior $\lambda \sim \mathcal{Gamma}\left(\alpha, \beta\right)$:

$$\lambda \mid \mathbf{X} \sim \mathcal{Gamma}\left(\alpha + \sum_{i=1}^{n} x_i,\ \beta + n\right).$$

Normal-Normal For c.i.i.d $\mathbf{X} = (X_1, \ldots, X_n)$, with each $X \mid \mu \sim \mathcal{N}(\mu, \sigma^2 = \frac{1}{\tau})$ with known precision $\tau > 0$ and prior $\mu \sim \mathcal{N}\left(m_0, \frac{1}{t_0}\right)$ and $t_0 > 0$:

$$\mu \mid \mathbf{X} \sim \mathcal{N}\left(m_1, \frac{1}{t_1}\right),$$

where

$$t_1 = t_0 + n\tau, \quad \text{and} \quad m_1 = \frac{t_0 m_0 + n\tau\bar{x}}{t_0 + n\tau} = \frac{t_0 m_0 + n\tau\bar{x}}{t_1}.$$

Normal-Gamma For c.i.i.d $\mathbf{X} = (X_1, \ldots, X_n)$, where each $X \sim \mathcal{N}(\mu, \frac{1}{\tau})$ and priors

$$\mu \mid \tau \sim \mathcal{N}\left(m_0, v_0^2 = \frac{1}{k_0\tau}\right) \quad \text{and} \quad \tau \sim \mathcal{Gamma}\left(\alpha_0, \beta_0\right),$$

with constant $k_0 > 0$:

$$\mu \mid \mathbf{X}, \tau \sim \mathcal{N}\left(m_1, \frac{1}{k_1\tau}\right) \quad \text{and} \quad \tau \mid \mathbf{X} \sim \mathcal{Gamma}\left(\alpha_1, \beta_1\right),$$

where

$$k_1 = k_0 + n, \quad m_1 = \frac{k_0 m_0 + n\bar{x}}{k_0 + n} = \frac{k_0 m_0 + n\bar{x}}{k_1},$$

$$\alpha_1 = \alpha_0 + \frac{n}{2}, \quad \beta_1 = \beta_0 + \frac{1}{2}(n-1)s^2 + \frac{k_0 n(\bar{x} - m_0)^2}{2k_1}.$$

Jeffreys prior:

Scalar θ*:* $f(\theta) \propto \sqrt{\mathcal{I}(\theta)}$
Vector $\boldsymbol{\theta}$*:* $f(\boldsymbol{\theta}) \propto \sqrt{\det\left(\mathcal{I}(\boldsymbol{\theta})\right)}$

Bayesian Inference

Large sample posterior: For $X = (X_1, \ldots, X_n)$ c.i.i.d with regular likelihood $f(x \mid \theta)$ and n "large enough", the posterior distribution for $\theta \mid x$ is approximately

$$f(\boldsymbol{\theta} \mid \mathbf{x}) \mathrel{\dot\sim} \mathcal{N}_p\left(\hat{\boldsymbol{\theta}}, I\left(\hat{\boldsymbol{\theta}}\right)^{-1}\right),$$

Predictive Distributions *Prior predictive:*

$$f(\mathbf{x}) = \int f(\mathbf{x}, \boldsymbol{\theta})\ \mathrm{d}\boldsymbol{\theta} = \int f(\mathbf{x} \mid \boldsymbol{\theta})\, f(\boldsymbol{\theta})\ \mathrm{d}\boldsymbol{\theta}$$

Posterior predictive:

$$f(\mathbf{x}^* \mid \mathbf{x}) = \int f(\mathbf{x}^*, \boldsymbol{\theta} \mid \mathbf{x})\ \mathrm{d}\boldsymbol{\theta} = \int f(\mathbf{x}^* \mid \boldsymbol{\theta}, \mathbf{x})\, f(\boldsymbol{\theta} \mid \mathbf{x})\ \mathrm{d}\boldsymbol{\theta}$$

Exponential Family

Probability density function The pdf of $\mathbf{X}$ with vector parameter $\boldsymbol{\theta}$ belongs to the q-parameter exponential family of distributions if it can be written in the form

$$f(\mathbf{x} \mid \boldsymbol{\theta}) = b(\mathbf{x}) \exp\left\{\boldsymbol{\phi}(\boldsymbol{\theta})^T \mathbf{t}(\mathbf{x}) - a(\boldsymbol{\phi})\right\}.$$

Bayes Factors

Bayes factors in favour of null hypothesis Simple vs. simple comparisons:

$$B_{01} = \frac{f(\mathbf{x} \mid \theta = \theta_0)}{f(\mathbf{x} \mid \theta = \theta_1)},$$

Composite vs. composite comparisons:

$$B_{01} = \frac{\displaystyle\int_{\theta \in \Omega_0} f(\mathbf{x} \mid \theta)\, f_0(\theta)\, \mathrm{d}\theta}{\displaystyle\int_{\theta \in \Omega_1} f(\mathbf{x} \mid \theta)\, f_1(\theta)\, \mathrm{d}\theta}.$$

Simple vs. composite comparisons:

$$B_{01} = \frac{f(\mathbf{x} \mid \theta = \theta_0)}{\displaystyle\int_{\theta \in \Omega} f(\mathbf{x} \mid \theta)\, f_1(\theta)\, \mathrm{d}\theta}.$$

Posterior probability of null hypothesis

$$p_0 = \left[1 + \frac{1 - \pi_0}{\pi_0} B_{01}^{-1}\right]^{-1}$$

Interpretation of Bayes Factors

$2\log_e(B_{10})$	B_{10}	Evidence against $\mathcal{H}_0$
$(-\infty, 0)$	$(0, 1)$	None
$(0, 2)$	$(1, 3)$	Not worth more than a bare mention
$(2, 6)$	$(3, 20)$	Positive
$(6, 10)$	$(20, 150)$	Strong
> 10	> 150	Very strong

Normal approximations to Wilcoxon's test statistics

Rank-sum test statistic For n, m large,

$$W \mathrel{\dot\sim} \mathcal{N}\left(\mu_W, \sigma_W^2\right),$$

where

$$\mu_W = \frac{1}{2}n(n + m + 1), \qquad \sigma_W^2 = \frac{1}{12}nm(n + m + 1).$$

Signed-rank test statistic For large n,

$$V^+ \mathrel{\dot\sim} \mathcal{N}\left(\mu_V, \sigma_V^2\right),$$

where

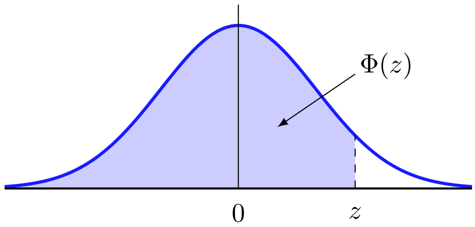
$$\mu_V = \frac{1}{4}n(n + 1), \qquad \sigma_V^2 = \frac{1}{24}n(n + 1)(2n + 1),$$

and V^+ is the sum of the ranks of the absolute differences for all pairs with positive difference. The same result applies to V^- (sum of the ranks of the absolute differences for all pairs with negative difference).

Table A: Probabilities for the standard normal distribution

Table entry for z is the probability lying to the left of z for a $N(0,1)$ distribution.

$$\Phi(z) = P[Z < z]$$

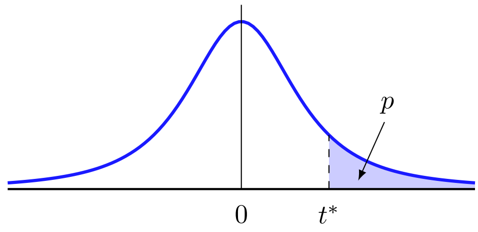


$N(0,1)$ density

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.40	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.50	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.60	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.70	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.80	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.00	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.20	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.30	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.40	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.50	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.60	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.70	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.80	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.90	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.00	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.10	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.20	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.30	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.40	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Table B: Probabilities for the t-distribution

Table entry for p and C is the critical value t^* with probability p lying to its right and probability C lying between $-t^*$ and t^* for a t-distribution with k degrees of freedom.

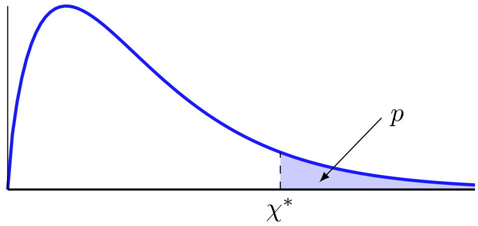


t_k density

	Upper-tail probability p											
k	0.25	0.2	0.15	0.1	0.05	0.025	0.02	0.01	0.005	0.0025	0.001	0.0005
1	1.000	1.376	1.963	3.078	6.314	12.706	15.895	31.821	63.657	127.321	318.309	636.619
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.089	22.327	31.599
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.215	12.924
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
70	0.678	0.847	1.044	1.294	1.667	1.994	2.093	2.381	2.648	2.899	3.211	3.435
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.090	3.291
	50%	60%	70%	80%	90%	95%	96%	98%	99%	99.5%	99.8%	99.9%
	Confidence level C											

Table C: Probabilities for the χ^2-distribution

Table entry for p is the point χ^* with probability p lying above it for a χ^2 distribution with k degrees of freedom.



χ^2_k density

	Upper-tail probability p											
k	0.995	0.99	0.975	0.95	0.9	0.75	0.25	0.1	0.05	0.025	0.01	0.005
1	0.000	0.000	0.001	0.004	0.016	0.102	1.323	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	0.575	2.773	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	1.213	4.108	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	1.923	5.385	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	2.675	6.626	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	3.455	7.841	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	4.255	9.037	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	5.071	10.219	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	5.899	11.389	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	6.737	12.549	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	7.584	13.701	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	8.438	14.845	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	9.299	15.984	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	10.165	17.117	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	11.037	18.245	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	11.912	19.369	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	12.792	20.489	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	13.675	21.605	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	14.562	22.718	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	15.452	23.828	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	16.344	24.935	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	17.240	26.039	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	18.137	27.141	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	19.037	28.241	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	19.939	29.339	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	20.843	30.435	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	21.749	31.528	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	22.657	32.620	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	23.567	33.711	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	24.478	34.800	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	33.660	45.616	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	42.942	56.334	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	52.294	66.981	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	61.698	77.577	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	71.145	88.130	96.578	101.879	106.629	112.329	116.321
100	67.328	70.065	74.222	77.929	82.358	90.133	109.141	118.498	124.342	129.561	135.807	140.169
	0.005	0.01	0.025	0.05	0.1	0.25	0.75	0.9	0.95	0.975	0.99	0.995
	Lower-tail probability α											

Table D: Critical values for the Rank-sum Test

Reject the hypothesis of identical populations in a two-sided test at level α if the test statistic W from a group of size n is *less than* the value T_L shown in the following table, or greater than the value T_U where

$$T_U = n(n + m + 1) - T_L.$$

		m								
$\alpha = 0.01$		2	3	4	5	6	7	8	9	10
	2	-	-	-	-	-	-	-	-	-
	3	-	-	-	-	-	-	-	7	7
	4	-	-	-	-	11	11	12	12	13
	5	-	-	-	16	17	17	18	19	20
n	6	-	-	22	23	24	25	26	27	28
	7	-	-	29	30	32	33	35	36	38
	8	-	-	38	39	41	43	44	46	48
	9	-	46	47	49	51	53	55	57	59
	10	-	56	58	60	62	65	67	69	72

		m								
$\alpha = 0.05$		2	3	4	5	6	7	8	9	10
	2	-	-	-	-	-	-	4	4	4
	3	-	-	-	7	8	8	9	9	10
	4	-	-	11	12	13	14	15	15	16
	5	-	16	17	18	19	21	22	23	24
n	6	-	23	24	25	27	28	30	32	33
	7	-	30	32	34	35	37	39	41	43
	8	37	39	41	43	45	47	50	52	54
	9	46	48	50	53	56	58	61	63	66
	10	56	59	61	64	67	70	73	76	79

		m								
$\alpha = 0.10$		2	3	4	5	6	7	8	9	10
	2	-	-	-	4	4	4	5	5	5
	3	-	7	7	8	9	9	10	11	11
	4	-	11	12	13	14	15	16	17	18
	5	16	17	18	20	21	22	24	25	27
n	6	22	24	25	27	29	30	32	34	36
	7	29	31	33	35	37	40	42	44	46
	8	38	40	42	45	47	50	52	55	57
	9	47	50	52	55	58	61	64	67	70
	10	57	60	63	67	70	73	76	80	83

Table E: Critical values for the Signed-rank Test

Reject the null hypothesis of identical populations if the test statistic V is *less than* the value T shown in the following table.

Sample size	Level of significance for a two-sided test					
n	20%	10%	5%	2%	1%	0.5%
	10%	5%	2.5%	1%	0.5%	0.25%
	Level of significance for a one-sided test					
5	3	1	-	-	-	-
6	4	3	1	-	-	-
7	6	4	3	1	-	-
8	9	6	4	2	1	-
9	11	9	6	4	2	1
10	15	11	9	6	4	2
11	18	14	11	8	6	4
12	22	18	14	10	8	6
13	27	22	18	13	10	8
14	32	26	22	16	13	10
15	37	31	26	20	16	13
16	43	36	30	24	20	16
17	49	42	35	28	24	20
18	56	48	41	33	28	24
19	63	54	47	38	33	28
20	70	61	53	44	38	33
21	78	68	59	50	43	38
22	87	76	66	56	49	43
23	95	84	74	63	55	49
24	105	92	82	70	62	55
25	114	101	90	77	69	61
26	125	111	99	85	76	68
27	135	120	108	93	84	75
28	146	131	117	102	92	83
29	158	141	127	111	101	91
30	170	152	138	121	110	99