

EXAMINATION PAPER

Examination Session: May/June

Year: 2025

Exam Code:

MATH2727-WE01-SP

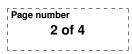
Title:

Topology II (2023/24 Syllabus)

Time:	2 hours	
Additional Material provided:		
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Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators
		is forbidden.

Instructions to Candidates:	Answer all questions.	
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.	
	Write your answer in the white-covered answer booklet with barcodes.	
	Begin your answer to each question on a new page.	

Revision:



In these questions \mathbb{R} , \mathbb{R}^2 and \mathbb{R}^3 always have the standard topology.

SECTION A

Q1 In \mathbb{R}^3 , let

 $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 4\}, \quad T = \{(x, y, z) \in \mathbb{R}^3 \mid |x - y| < 1\}.$

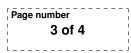
- (a) For each of the sets S and T, determine whether it is open, whether it is closed, and whether it is bounded. Justify your answers.
- (b) Now consider the sets $S \cup T$, $S \cap T$, $S \setminus T$ and $T \setminus S$. Find one of these sets which is closed, and one which is open, and explain how you know this.
- (c) Of the six sets considered above, which are compact? Justify your answer.
- **Q2** (a) Let the set $X = \{a, b, c\}$. Which one of the following subsets of $\mathcal{P}(X)$ is a topology on X? Why are the others not topologies?

$$\tau_1 = \{ \emptyset, \{a\}, X\} \quad \tau_2 = \{ \emptyset, \{a\}, \{a, b\} \} \quad \tau_3 = \{ \emptyset, \{a, b\}, \{a, c\}, X \}$$

(b) Let the set $Y = \{1, 2, 3, 4\}$. Which one of the following subsets of $\mathcal{P}(Y)$ is a topology on Y? Why are the others not topologies?

 $\tau_4 = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, Y\} \quad \tau_5 = \{\emptyset, \{2, 3\}, Y\} \quad \tau_6 = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, Y\}$

- (c) We rename the topologies that you found τ_X and τ_Y respectively. Using these, construct the product topology $\tau_{X \times Y}$ for the product space $X \times Y$. For clarity, you should give at least some of the open sets explicitly, as lists of elements.
- (d) In general, let topological spaces W and Z have finite topologies, so $|\tau_W| = m, |\tau_Z| = n$. Show that $|\tau_{W \times Z}| \ge (m-1)(n-1) + 1$.



Then X is connected.

SECTION B

- **Q3** (a) Write down the definitions of **connectedness** and **path-connectedness** for a general topological space X.
 - (b) Show that a path-connected topological space X is necessarily connected. You may use without proving the following statement from the lectures: Let $\mathcal{A} = \{A_j \mid j \in J\}$ be a collection of subsets of X such that $\bigcup_{j \in J} A_j = X$. Assume that each A_j is connected and for each $j, k \in J$ we have $A_j \cap A_k \neq \emptyset$.

In the following, I denotes the interval $I = [0, 2\pi] \subseteq \mathbb{R}$. We now consider two families of maps, $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$, where

$$f_n: I \to \mathbb{R}^2, \ f_n(x) = n(\cos x, \sin x),$$

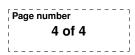
 $g_n: I \to \mathbb{R}^2, \ g_n(x) = \left(n + \frac{x}{2\pi}\right)(\cos x, \sin x)$

(c) Draw two sketches showing the images $f_n(I)$ and $g_n(I)$ respectively, for n = 1, 2, 3. Are the f_n homeomorphisms onto their images? Are the g_n homeomorphisms onto their images? Explain briefly.

Let

$$F = \bigcup_{n=1}^{\infty} f_n(I), \quad G = \bigcup_{n=1}^{\infty} g_n(I).$$

- (d) Write down the definition of **compactness** for a general topological space X. Use this definition to show that neither F nor G is compact.
- (e) Is F connected? Is G connected? Justify your answer.





Q4 In \mathbb{R}^2 , let

$$L_1 = \{(x, 1) \mid -1 \le x \le 1\}, \quad L_2 = \{(x, 2) \mid -1 \le x \le 1\}, \text{ and } X = L_1 \cup L_2.$$

We shall construct three different quotient spaces from X. In each case an equivalence \sim is given, and we consider the quotient space X/\sim with the quotient topology. You may wish to mention the quotient map $\pi: X \to X/\sim$, and write $\pi(x, y) = [(x, y)]$.

(a) $(x_1, y_1) \sim_a (x_2, y_2) \iff x_1 = x_2, y_1 = y_2$ OR $x_1 = x_2 = 0$

Show that X/\sim_a is homeomorphic to

 $A = \{(x,0) \mid |x| \le 1\} \cup \{(0,y) \mid |y| \le 1\} \subseteq \mathbb{R}^2.$

Start by giving an explicit map $f: X \to A$.

(b) $(x_1, y_1) \sim_b (x_2, y_2) \iff x_1 = x_2, y_1 = y_2$ OR $x_1 = x_2 = -1$ OR $x_1 = x_2 = 1$ Give a familiar subset B of \mathbb{R}^2 which is homeomorphic to X/\sim_b . Specify a

Give a familiar subset B of \mathbb{R}^2 which is homeomorphic to X/\sim_b . Specify a homeomorphism $g: X/\sim_b \to B$, and prove that it is one. (You may refer to part (a) if some of the argument is the same.)

(c) $(x_1, y_1) \sim_c (x_2, y_2) \iff x_1 = x_2, y_1 = y_2 \text{ OR } x_1 = x_2 \neq 0$

Show that X/\sim_c is not Hausdorff.