Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH2727-WE01

Title:
Topology II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

In these questions $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ always have the standard topology.

SECTION A

Q1 In $\mathbb{R}^3$, let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 4\}, \quad T = \{(x, y, z) \in \mathbb{R}^3 \mid |x - y| < 1\}.$$

(a) For each of the sets S and T, determine whether it is open, whether it is closed, and whether it is bounded. Justify your answers.

(b) Now consider the sets $S \cup T$, $S \cap T$, $S \setminus T$ and $T \setminus S$. Show that one of these sets is closed, and one of them is open. (No need to show that others are not.)

(c) Of the six sets considered above, which are compact? Justify your answer.

Q2 (a) Let the set $X = \{a, b, c\}$. Which one of the following subsets of $\mathcal{P}(X)$ is a topology on X? Why are the others not topologies?

$$\tau_1 = \{\emptyset, \{a\}, X\} \quad \tau_2 = \{\emptyset, \{a\}, \{a, b\}\} \quad \tau_3 = \{\emptyset, \{a, b\}, \{a, c\}, X\}$$

(b) Let the set $Y = \{1, 2, 3, 4\}$. Which one of the following subsets of $\mathcal{P}(Y)$ is a topology on Y? Why are the others not topologies?

$$\tau_4 = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, Y\} \quad \tau_5 = \{\emptyset, \{2, 3\}, Y\} \quad \tau_6 = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, Y\}$$

(c) We rename the topologies that you found τ_X and τ_Y respectively. Using these, construct the product topology $\tau_{X\times Y}$ for the product space $X \times Y$. For clarity, you should give at least some of the open sets explicitly, as lists of elements.

(d) In general, let topological spaces W and Z have finite topologies, so $|\tau_W| = m, |\tau_Z| = n$. Show that $|\tau_{W\times Z}| \geq (m - 1)(n - 1) + 1$.

SECTION B

Q3 The interval $I = [0, 2\pi] \subseteq \mathbb{R}$ is connected. By answering questions (a), (b), and (c) below, complete the following proof by contradiction of this fact.

Suppose I is not connected. Then there are open sets U and V in $\mathbb{R}$, such that $I \subseteq U \cup V$, $U \cap V = \emptyset$, and both $U \cap I$ and $V \cap I$ are non-empty, so we can find $x \in U \cap I$, $y \in V \cap I$. Since $x \neq y$, without loss of generality assume $x < y$. Let

$$A = \{a \in [x, \infty) \mid [x, a] \subseteq U\}.$$

(a) Explain why A must have a supremum, $s \in I$.

(b) Show that s is not in U.

(c) Show that s is not in V.

Now we consider two families of functions, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$, where

$$f_n : I \to \mathbb{R}^2, \quad f_n(x) = n(\cos x, \sin x),$$

$$g_n : I \to \mathbb{R}^2, \quad g_n(x) = \left(n + \frac{x}{2\pi}\right)(\cos x, \sin x).$$

(d) Draw two sketches showing the images $f_n(I)$ and $g_n(I)$ respectively, for $n = 1, 2, 3$. Are the f_n homeomorphisms onto their images? Are the g_n homeomorphisms onto their images? Explain briefly.

Let

$$F = \bigcup_{n=1}^{\infty} f_n(I), \quad G = \bigcup_{n=1}^{\infty} g_n(I).$$

(e) Write down the definition of compactness for a general topological space X. Use this definition to show that neither F nor G is compact.

(f) Is F connected? Is G connected? If "no", show this. If "yes", describe briefly a way to show this, but full detail is not required.

Q4 In $\mathbb{R}^2$, let

$$L_1 = \{(x, 1) \mid -1 \leq x \leq 1\}, \quad L_2 = \{(x, 2) \mid -1 \leq x \leq 1\}, \text{ and } X = L_1 \cup L_2.$$

We shall construct three different quotient spaces from X.
In each case an equivalence $\sim$ is given, and we consider the quotient space $X/\sim$ with the quotient topology. You may wish to mention the quotient map $\pi : X \to X/\sim$, and write $\pi((x, y)) = [(x, y)]$.

(a) $(x_1, y_1) \sim_a (x_2, y_2) \iff x_1 = x_2,\ y_1 = y_2$ OR $x_1 = x_2 = 0$

Show that $X/\sim_a$ is homeomorphic to

$$A = \{(x, 0) \mid |x| \leq 1\} \cup \{(0, y) \mid |y| \leq 1\} \subseteq \mathbb{R}^2.$$

Start by giving an explicit map $f : X \to A$.

(b) $(x_1, y_1) \sim_b (x_2, y_2) \iff x_1 = x_2,\ y_1 = y_2$ OR $x_1 = x_2 = -1$ OR $x_1 = x_2 = 1$

Give a familiar subset B of $\mathbb{R}^2$ which is homeomorphic to $X/\sim_b$. Specify a homeomorphism $\bar{g} : X/\sim_b \to B$, and prove that it is one. (You may refer to part (a) if some of the argument is the same.)

(c) $(x_1, y_1) \sim_c (x_2, y_2) \iff x_1 = x_2,\ y_1 = y_2$ OR $x_1 = x_2 \neq 0$

Show that $X/\sim_c$ is not Hausdorff.