

## **EXAMINATION PAPER**

Examination Session: May/June

2025

Year:

Exam Code:

MATH2727-WE01

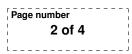
Title:

Topology II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions.
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.
	Write your answer in the white-covered answer booklet with barcodes.
	Begin your answer to each question on a new page.

**Revision:** 



In these questions  $\mathbb{R}$ ,  $\mathbb{R}^2$  and  $\mathbb{R}^3$  always have the standard topology.

## SECTION A

## **Q1** In $\mathbb{R}^3$ , let

 $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 4\}, \quad T = \{(x, y, z) \in \mathbb{R}^3 \mid |x - y| < 1\}.$ 

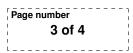
- (a) For each of the sets S and T, determine whether it is open, whether it is closed, and whether it is bounded. Justify your answers.
- (b) Now consider the sets  $S \cup T$ ,  $S \cap T$ ,  $S \setminus T$  and  $T \setminus S$ . Show that one of these sets is closed, and one of them is open. (No need to show that others are not.)
- (c) Of the six sets considered above, which are compact? Justify your answer.
- **Q2** (a) Let the set  $X = \{a, b, c\}$ . Which one of the following subsets of  $\mathcal{P}(X)$  is a topology on X? Why are the others not topologies?

$$\tau_1 = \{\emptyset, \{a\}, X\} \quad \tau_2 = \{\emptyset, \{a\}, \{a, b\}\} \quad \tau_3 = \{\emptyset, \{a, b\}, \{a, c\}, X\}$$

(b) Let the set  $Y = \{1, 2, 3, 4\}$ . Which one of the following subsets of  $\mathcal{P}(Y)$  is a topology on Y? Why are the others not topologies?

 $\tau_4 = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, Y\} \quad \tau_5 = \{\emptyset, \{2, 3\}, Y\} \quad \tau_6 = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, Y\}$ 

- (c) We rename the topologies that you found  $\tau_X$  and  $\tau_Y$  respectively. Using these, construct the product topology  $\tau_{X \times Y}$  for the product space  $X \times Y$ . For clarity, you should give at least some of the open sets explicitly, as lists of elements.
- (d) In general, let topological spaces W and Z have finite topologies, so  $|\tau_W| = m, |\tau_Z| = n$ . Show that  $|\tau_{W \times Z}| \ge (m-1)(n-1) + 1$ .



## SECTION B

Q3 The interval  $I = [0, 2\pi] \subseteq \mathbb{R}$  is connected. By answering questions (a), (b), and (c) below, complete the following proof by contradiction of this fact.

Suppose I is not connected. Then there are open sets U and V in  $\mathbb{R}$ , such that  $I \subseteq U \cup V$ ,  $U \cap V = \emptyset$ , and both  $U \cap I$  and  $V \cap I$  are non-empty, so we can find  $x \in U \cap I$ ,  $y \in V \cap I$ . Since  $x \neq y$ , without loss of generality assume x < y. Let

$$A = \{ a \in [x, \infty) \mid [x, a] \subseteq U \}.$$

- (a) Explain why A must have a supremum,  $s \in I$ .
- (b) Show that s is not in U.
- (c) Show that s is not in V.

Now we consider two families of functions,  $\{f_n\}_{n\in\mathbb{N}}$  and  $\{g_n\}_{n\in\mathbb{N}}$ , where

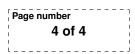
$$f_n: I \to \mathbb{R}^2, \quad f_n(x) = n(\cos x, \sin x),$$
$$g_n: I \to \mathbb{R}^2, \quad g_n(x) = \left(n + \frac{x}{2\pi}\right)(\cos x, \sin x).$$

(d) Draw two sketches showing the images  $f_n(I)$  and  $g_n(I)$  respectively, for n = 1, 2, 3. Are the  $f_n$  homeomorphisms onto their images? Are the  $g_n$  homeomorphisms onto their images? Explain briefly.

Let

$$F = \bigcup_{n=1}^{\infty} f_n(I), \quad G = \bigcup_{n=1}^{\infty} g_n(I).$$

- (e) Write down the definition of compactness for a general topological space X. Use this definition to show that neither F nor G is compact.
- (f) Is F connected? Is G connected? If "no", show this. If "yes", describe briefly a way to show this, but full detail is not required.





Q4 In  $\mathbb{R}^2$ , let

$$L_1 = \{(x, 1) \mid -1 \le x \le 1\}, \quad L_2 = \{(x, 2) \mid -1 \le x \le 1\}, \text{ and } X = L_1 \cup L_2.$$

We shall construct three different quotient spaces from X. In each case an equivalence  $\sim$  is given, and we consider the quotient space  $X/\sim$  with the quotient topology. You may wish to mention the quotient map  $\pi: X \to X/\sim$ , and write  $\pi((x, y)) = [(x, y)]$ .

(a)  $(x_1, y_1) \sim_a (x_2, y_2) \iff x_1 = x_2, y_1 = y_2$  OR  $x_1 = x_2 = 0$ 

Show that  $X/\sim_a$  is homeomorphic to

 $A = \{(x,0) \mid |x| \le 1\} \cup \{(0,y) \mid |y| \le 1\} \subseteq \mathbb{R}^2.$ 

Start by giving an explicit map  $f: X \to A$ .

(b)  $(x_1, y_1) \sim_b (x_2, y_2) \iff x_1 = x_2, y_1 = y_2$  OR  $x_1 = x_2 = -1$  OR  $x_1 = x_2 = 1$ 

Give a familiar subset B of  $\mathbb{R}^2$  which is homeomorphic to  $X/\sim_b$ . Specify a homeomorphism  $\bar{g}: X/\sim_b \to B$ , and prove that it is one. (You may refer to part (a) if some of the argument is the same.)

(c)  $(x_1, y_1) \sim_c (x_2, y_2) \iff x_1 = x_2, y_1 = y_2$  OR  $x_1 = x_2 \neq 0$ 

Show that  $X/\sim_c$  is not Hausdorff.