EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH3021-WE01

Title: Differential Geometry III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Let $\boldsymbol{\alpha} : (-1, 1) \to \mathbb{R}^3$,

$$\boldsymbol{\alpha}(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}} \right).$$

Show that $\boldsymbol{\alpha}$ is a unit speed curve and compute its curvature and torsion.

Q2 Let

$$\boldsymbol{x}(u, v) = (f(v)\cos(u), f(v)\sin(u), g(v))$$

with $u \in (0, 2\pi)$ and $f, g : (a, b) \to \mathbb{R}$ be a local parametrisation of a surface of revolution.

(a) Compute the coefficients of the first fundamental form with respect to $\boldsymbol{x}$. Which relation do the functions f, g need to satisfy in order that $\boldsymbol{x}$ is an isothermal parametrisation?

(b) Assume that $f(v) = \cosh(v)$. Find all functions g for which $\boldsymbol{x}$ is an isothermal parametrisation.

Q3 Consider the parametrized surface $\boldsymbol{x} : U = (0, 1) \times (0, 1) \to \mathbb{R}^3$,

$$\boldsymbol{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right).$$

(a) Calculate the coefficients of the first fundamental form with respect to $\boldsymbol{x}$.

(b) Calculate a Gauss map $\boldsymbol{N} : \boldsymbol{x}(U) \to S^2$.

Q4 (a) Define what it means that a curve on a parametrized surface is a geodesic.

(b) Prove or disprove by counterexample the following claim : "If a curve on a surface has constant speed, then it is a geodesic".

(c) Consider the parametrized surface $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$,

$$\boldsymbol{x}(u, v) = \left(u - v, u + v, 4u^2 - 4v^2 \right).$$

Prove or disprove that the curve $u \mapsto \boldsymbol{x}(u, u)$, $u \in \mathbb{R}$ on the parametrized surface $\boldsymbol{x}$ is a geodesic.

SECTION B

Q5 Let $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^3$ be a smooth unit speed curve with nowhere vanishing curvature $\kappa_{\boldsymbol{\alpha}} : [a, b] \to \mathbb{R}$ and $\boldsymbol{t}_{\boldsymbol{\alpha}}, \boldsymbol{n}_{\boldsymbol{\alpha}}, \boldsymbol{b}_{\boldsymbol{\alpha}}$ be the moving frame associated to $\boldsymbol{\alpha}$ and $\tau_{\boldsymbol{\alpha}} : [a, b] \to \mathbb{R}$ the torsion of $\boldsymbol{\alpha}$. Let $\boldsymbol{\beta} : [a, b] \to \mathbb{R}^3$ be given by

$$\boldsymbol{\beta}(s) := \boldsymbol{\alpha}(s) + \cos(2s)\boldsymbol{n}_{\boldsymbol{\alpha}}(s) + \sin(2s)\boldsymbol{b}_{\boldsymbol{\alpha}}(s).$$

(a) State the Serret-Frenet formulae for the curve $\boldsymbol{\alpha}$ and compute $\|\boldsymbol{\beta}'(s)\|$ in terms of $\kappa_{\boldsymbol{\alpha}}$ and $\tau_{\boldsymbol{\alpha}}$.

(b) Assume that $[a, b] = [0, \frac{1}{2}]$, $\kappa_{\boldsymbol{\alpha}}(s) = \frac{4}{\cos(2s)}$ and $\tau_{\boldsymbol{\alpha}}(s) = -2$ for all $s \in [0, \frac{1}{2}]$. Compute the length $L(\boldsymbol{\beta})$ of $\boldsymbol{\beta} : [0, \frac{1}{2}] \to \mathbb{R}^3$.

(c) Assume that $\kappa_{\boldsymbol{\alpha}} : [a, b] \to \mathbb{R} \setminus \{0\}$ is arbitrary and $\tau_{\boldsymbol{\alpha}}(s) = 2$ for all $s \in [a, b]$. Compute the curvature $\kappa_{\boldsymbol{\beta}} : [a, b] \to \mathbb{R}$ of the curve $\boldsymbol{\beta}$ in terms of the curvature $\kappa_{\boldsymbol{\alpha}}$.

Q6 For part (b), we use the following models of the hyperbolic plane:

Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} : \mathrm{Im}(z) = y > 0\}$ be the *upper half plane model* with associated bilinear form

$$\langle w_1, w_2 \rangle_z = \frac{\mathrm{Re}(w_1 \overline{w}_2)}{y^2}$$

for $w_1, w_2 \in \mathbb{C}$, where the tangent space $T_z\mathbb{H}$ is canonically identified with $\mathbb{C}$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the *Poincaré unit disk model* with associated bilinear form

$$\langle w_1, w_2 \rangle_z = \frac{4\mathrm{Re}(w_1 \overline{w}_2)}{(1 - |z|^2)^2}$$

for $w_1, w_2 \in \mathbb{C}$, where the tangent space $T_z\mathbb{D}$ is canonically identified with $\mathbb{C}$.

(a) Show the parallelogram equality for a general surface S with first fundamental form $I_p(w) = I_p^S(w) = \langle w, w \rangle_p$, that is,

$$\langle w_1, w_2 \rangle_p = \frac{I_p(w_1 + w_2) - I_p(w_1) - I_p(w_2)}{2}.$$

(b) Show that the Cayley map $f : \mathbb{H} \to \mathbb{D}$, defined by

$$f(z) = \frac{z - i}{z + i},$$

is a global isometry, that is

$$I^{\mathbb{D}}_{f(z)}(d_z f(w)) = I^{\mathbb{H}}_z(w)$$

for all $z \in \mathbb{H}$ and $w \in T_z\mathbb{H}$. You can use without proof that f is a biholomorphic map from $\mathbb{H}$ to $\mathbb{D}$ and therefore a diffeomorphism.

Q7 Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a global parametrisation of a regular surface S and $\boldsymbol{N} : U \to S^2$ be a unit length normal, that is $\boldsymbol{N}(u,v) \perp T_{x(u,v)}S$ and $\|\boldsymbol{N}(u,v)\| = 1$ for all $(u,v) \in U$. For every $t \in \mathbb{R}$, we consider the set $S_t \subset \mathbb{R}^3$, defined as the image of the global parametrisation

$$\boldsymbol{y}(u,v) = \boldsymbol{x}(u,v) + t\boldsymbol{N}(u,v).$$

We will assume that, for $|t| > 0$ small, the sets S_t are again regular surfaces.

(a) Prove the identity

$$\frac{\partial \boldsymbol{N}}{\partial u} \times \frac{\partial \boldsymbol{N}}{\partial v} = K\, \boldsymbol{x}_u \times \boldsymbol{x}_v,$$

where K is the Gauss curvature of S, viewed as a function on U.

(b) Let E, F, G be the coefficients of the first fundamental form of $\boldsymbol{y}$. Express

$$\left.\frac{\partial E}{\partial t}\right|_{t=0}, \quad \left.\frac{\partial F}{\partial t}\right|_{t=0}, \quad \left.\frac{\partial G}{\partial t}\right|_{t=0}$$

in terms of the coefficients of the second fundamental form of $\boldsymbol{x}$.

(c) Show that

$$\boldsymbol{y}_u \times \boldsymbol{y}_v = (1 - 2Ht + Kt^2)\boldsymbol{x}_u \times \boldsymbol{x}_v,$$

where H is the mean curvature of the surface S, viewed as a function on U.

(d) Assume that the Gauss curvature K and the mean curvature H of S are globally bounded. Express, for small $|t| > 0$, the Gauss curvature of the regular surface S_t in terms of t, the Gauss curvature K and the mean curvature H of S.

Q8 Consider the upper half plane model $\mathbb{H} = \{(u,v) \in \mathbb{R}^2 : v > 0\}$ of the hyperbolic plane, with first fundamental form given by

$$E(u,v) = \frac{1}{v^2}, \quad F(u,v) = 0, \quad G(u,v) = \frac{1}{v^2}.$$

(a) Let $a < b$ and $c > 0$. Calculate the hyperbolic area of the subset

$$R := \{(u,v) \in \mathbb{H} : a \leq u \leq b, v \geq c\}.$$

(b) Calculate the geodesic curvature of the curve $\boldsymbol{\gamma} : [a,b] \to \mathbb{H}$, $\boldsymbol{\gamma}(t) = (t,c)$ for some constant $c > 0$.

(c) State the Gauss-Bonnet Theorem and explain all involved terms. Using the fact that $\mathbb{H}$ has constant Gauss curvature -1 (without proof), verify in full detail that the Gauss-Bonnet Theorem holds for the set R.