EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH3041-WE01

Title:
Galois Theory III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 (a) Find the minimal polynomials of $\theta = \sqrt{5} + i$ over $\mathbb{Q}(\sqrt{5})$ and over $\mathbb{Q}(i)$.

(b) Given $\theta = \sqrt{5} + i$, express $\sqrt{5}$ and i in the form $a + b\theta + c\theta^2 + d\theta^3$ where $a, b, c, d \in \mathbb{Q}$, and find the minimal polynomial of θ over $\mathbb{Q}$.

Q2 (a) By considering automorphisms in a suitable Galois group or otherwise, prove that $\sqrt[3]{7} \notin \mathbb{Q}(\sqrt[3]{2})$.

(b) Give an explicit basis of the field $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{7})$ as a vector space over $\mathbb{Q}$.

Q3 (a) Find the Galois group of the cubic polynomial $x^3 + 4x - 4$ over $\mathbb{Q}$.

(b) Find the Galois group of the quartic polynomial $x^4 + 3x - 3$ over $\mathbb{Q}$.

Q4 Let $\zeta \in \mathbb{C}$ be an 85th primitive root of unity and set $L = \mathbb{Q}(\zeta)$.

(a) Find the degree $[L : \mathbb{Q}]$ and describe the structure of $\mathrm{Gal}(L/\mathbb{Q})$ as a direct product of cyclic groups.

(b) How many distinct subfields $K \subset L$ are there with $[K : \mathbb{Q}] = 2$?

SECTION B

Q5 (i) Find all complex roots of the polynomial $x^4 - x\sqrt{6} + 7/4$.

(ii) Suppose that α and β are algebraic over $\mathbb{Q}$ and their minimal polynomials have degrees m and n respectively.

(a) Prove that in general we have $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] \leq mn$. Give an example of α and β with $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] \neq mn$.

(b) Suppose further that $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) = \mathbb{Q}$ and $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a normal extension. Prove that in this case we have $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = mn$.

Q6 Given a primitive 17th root of unity $\zeta \in \mathbb{C}$, define fields

$$K = \mathbb{Q}(\theta) \text{ where } \theta = \zeta + \zeta^2 + \zeta^4 + \zeta^8 + \zeta^{-1} + \zeta^{-2} + \zeta^{-4} + \zeta^{-8},$$
$$L = \mathbb{Q}(\mu) \text{ where } \mu = \zeta + \zeta^4 + \zeta^{-1} + \zeta^{-4}.$$

(a) Show that the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is generated by the automorphism σ defined by $\sigma(\zeta) = \zeta^6$.

(b) Show that $\theta = (-1 \pm \sqrt{17})/2$.

(c) Show that $\theta \in L$.

(d) Find the minimal polynomial of μ over K.

Q7 (i) Let $K = \mathbb{F}_2(\theta)$ where θ is a root of $f(x) = x^4 + x + 1 \in \mathbb{F}_2[x]$.

(a) Prove that $[K : \mathbb{F}_2] = 4$ and explain why the polynomials

$$g(x) = x^4 + x^3 + 1 \qquad \text{and} \qquad h(x) = x^4 + x^3 + x^2 + x + 1$$

both split completely in $K[x]$.

(b) Find all roots of $f(x)$, $g(x)$ and $h(x)$ in K as linear combinations of the $\mathbb{F}_2$-basis $\{1, \theta, \theta^2, \theta^3\}$ of K.

(ii) How many distinct irreducible polynomials of degree 12 are there in $\mathbb{F}_2[x]$?

Q8 Let $d = 2 + \sqrt{3}$ and $\theta = \sqrt{2 + \sqrt{d}}$, and define $L = K(\theta)$ where $K = \mathbb{Q}(\sqrt{3})$.

(a) Prove that $[L : K] = 4$ and explain why L/K is a cyclic Galois extension.

(b) Suppose that σ is a generator of $\mathrm{Gal}(L/K)$. Prove that

$$\sigma(\theta) = \pm\frac{1}{\theta\sqrt{d}} \qquad \text{and} \qquad \sigma^2(\theta) = -\theta.$$

(c) Explain why there exists $A \in K(i)$ such that $L(i) = K(i)(\sqrt[4]{A})$.
(*Here,* $i = \sqrt{-1}$ *is a square root of* -1 *in* $\mathbb{C}$.)
Find an explicit value for A and prove that $L(i) = \mathbb{Q}(\zeta_{48})$ where ζ_{48} is a 48th primitive root of unity.