Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH30720-WE01

Title: Dynamical Systems V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Analyze the following dynamical system in Cartesian coordinates:

$$\begin{aligned} \dot{x} &= \frac{x \log(x^2+y^2)\left((x^2+y^2)-4\right)}{\sqrt{x^2+y^2}} - by \\ \dot{y} &= \frac{y \log(x^2+y^2)\left((x^2+y^2)-4\right)}{\sqrt{x^2+y^2}} + bx, \end{aligned}$$

where $b > 0$ is a constant.

(a) Rewrite this system in polar coordinates.

(b) Sketch the phase flow of the system in polar coordinates, and label all interesting points in the flow. Explain your reasoning for drawing that flow.

Q2 A linear two-dimensional dynamical system satifies equation $\dot{\mathbf{x}} = A\mathbf{x}$, where the matrix A is

$$A = \begin{pmatrix} 5/3 & 14/3 \\ 7/3 & -2/3 \end{pmatrix},$$

and $\mathbf{x}$ is a two dimensional state vector.

(a) Rewrite the system in the basis where the equations are decoupled, and solve it in that basis. Then using the explicit form of the similarity matrix Q, transform that solution into original coordinate system.

(b) Sketch the flow both in the basis where equations are decoupled and in the original basis, indicating what is the relation between the two flows.

Q3 Consider the 1-dimensional dynamical system

$$\dot{x} = x^2 + \mu x - x \ . \tag{1}$$

(a) For what values of x and μ does the system undergo a local bifurcation? Justify your answer.

(b) Draw the bifurcation diagram for this system. You do not need to specify what type of bifurcation it is.

(c) Add a small deformation ϵ such that the system becomes

$$\dot{x} = x^2 + \mu x - x + \epsilon \ . \tag{2}$$

Show that for $\epsilon < 0$ the system no longer undergoes a local bifurcation.

Q4 (a) Consider the 1-dimensional dynamical systems $\dot{x} = x - 5$ and $\dot{y} = 5$. Give an argument why they are not topologically conjugate.

(b) Consider the linear 2-dimensional dynamical systems

$$\dot{\boldsymbol{X}} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} \boldsymbol{X} \quad \text{and} \quad \dot{\boldsymbol{Y}} = \begin{pmatrix} 2+6\alpha & -8\alpha \\ 4\alpha & 2-6\alpha \end{pmatrix} \boldsymbol{Y} \ , \tag{3}$$

with $\alpha \in \mathbb{R} \setminus \{-1, 1\}$. Show that they are topologically conjugate without solving them.

SECTION B

Q5 In two dimensions, the dynamical system is defined as:

$$\dot{x} = y + 3x - y^3, \quad \dot{y} = -x - 3y + x^3.$$

(a) Find all the fixed points of this system. For each fixed point, determine its nature, including its type and stability.

(b) State what it means for a system to be Hamiltonian. Prove that if the system is Hamiltonian, then the Hamiltonian function is conserved along solutions of the system.

(c) Prove that the given system is Hamiltonian. Find the Hamiltonian H(x, y) explicitly.

(d) Sketch the level curves of the Hamiltonian ($H =$ constant) in the (x, y)-plane. Use these curves to illustrate the phase flow of the system.

Q6 If the equations of the two dimensional system are

$$\dot{x} = y, \quad \dot{y} = x(x+3)(x+4)\,,$$

answer the following questions:

(a) Solve the system explicitly to find the equations of the paths $y(x)$. In particular, focus on the path passing through the point $(0, 0)$ and path through the point $(-4, 0)$.

(b) Using the solution from the previous part and insights from the linearised analysis, sketch the phase flow for this system.

(c) State the stable manifold theorem for a generic 2D dynamical system.

(d) Apply the stable manifold theorem to this system and explicitly show that it holds by finding and describing the stable and unstable manifolds for the relevant fixed points.

Q7 (a) Consider the 3-dimensional dynamical system

$$\dot{x} = -\frac{\partial S}{\partial x}, \quad \dot{y} = -\frac{\partial S}{\partial y}, \quad \dot{z} = -\frac{\partial S}{\partial z}\ , \tag{4}$$

where S is a non-constant function $S : \mathbb{R}^3 \to \mathbb{R}$. Prove that this system cannot have a non-trivial periodic orbit, i.e. that all of its periodic orbits are just fixed points.

(b) Use the result above to show that the dynamical system

$$\dot{x} = \frac{2yz}{(xyz+2)^3}, \quad \dot{y} = \frac{2xz}{(xyz+2)^3}, \quad \dot{z} = \frac{2xy}{(xyz+2)^3}, \tag{5}$$

does not have any non-trivial periodic orbits.

(c) Consider now the system in part (a), with

$$S = (xy+1)^2 + (xz+1)^2 + (yz+1)^2\ . \tag{6}$$

Let C^3 be a cube with sides of length 2, centered at the origin. Denote the time evolution of all points in C^3, as dictated by the dynamical system, by $\varphi(t, C^3)$. Let $V(\varphi(t, C^3))$ be the volume of this region. Compute

$$\frac{dV(\varphi(t, C^3))}{dt} \tag{7}$$

at $t = 0$.

(d) Consider a 2-dimensional dynamical system:

$$\dot{x} = -\frac{\partial s}{\partial x}, \quad \dot{y} = -\frac{\partial s}{\partial y}\ , \tag{8}$$

where s is a function $s : \mathbb{R}^2 \to \mathbb{R}$. Give an example of s such that the system is Hamiltonian.

Q8 (a) Given a dynamical system and a point p in its phase space, define what is meant by *an ω-limit point* of p, and by *the ω-limit set* of p.

(b) Consider a 2-dimensional dynamical system

$$\dot{x} = F_1(x, y), \quad \dot{y} = F_2(x, y) \tag{9}$$

and the annulus $A_1 = \{(x, y) : 1 \leq (x-3)^2 + (y-3)^2 \leq 4\}$. Assume that $F_1(x, y)$ and $F_2(x, y)$ are chosen such that A_1 is positively invariant and contains at most finitely many critical points. Classify all possible ω-limit sets for all $p \in A_1$.

(c) Consider the dynamical system in (b), with

$$F_1(x, y) = (x-3)f(x, y) + y - 3, \quad F_2(x, y) = (y-3)f(x, y) - (x-3) \tag{10}$$

where $f(x, y)$ is an arbitrary function. Find the critical points of this system.

(d) Assume $f(x, y)$ in part (c) is chosen so that $A_1 = \{(x, y) : 1 \leq (x-3)^2 + (y-3)^2 \leq 4\}$ is positively invariant. What qualitative information can you infer about $\omega(p)$, with $p \in A_1$, for the system in part (c) ?

(e) Take the system in part (c), for the special case $f(x, y) = 0$. Consider the ellipse parametrised by $a(x-3)^2 + b(x-3)(y-3) + c(y-3)^2 = N$, where a, b, c, N are constants. Show that there exist a, b, c such that the ellipse is a periodic orbit for this dynamical system.