Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH30920-WE01

Title:
Mathematical Biology V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Starter for 10

Consider the following models for interacting populations $x(t)$ and $y(t)$:

$$\text{(i)}: \quad \frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x) + \frac{xy}{1+y}, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = y(1-y) + xy,$$

$$\text{(ii)}: \quad \frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x) - \frac{xy}{1+y}, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = y(1-y) + xy,$$

$$\text{(iii)}: \quad \frac{\mathrm{d}x}{\mathrm{d}t} = x - xy, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -y + xy,$$

$$\text{(iv)}: \quad \frac{\mathrm{d}x}{\mathrm{d}t} = a - x + x^2 y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = b - x^2 y.$$

State the physically valid equilibria in each of the models labelled (i)–(iv). For (iv) you should assume $a, b > 0$.

Q2 The one about the generic eigen-expansions for which I have no pun

Consider two models for population dispersal:

$$\frac{\partial u_1}{\partial t} = \nabla^2 u_1, \quad \frac{\partial^2 u_2}{\partial t^2} = \nabla^2 u_2,$$

for densities $u_1(x, y, t)$, $u_2(x, y, t)$, on a box domain with coordinates $(x, y) \in [0, L] \times [0, L]$ and no-flux boundary conditions.

(a) State the general solution to the equation for u_1 as a Fourier series.

(b) The solution for u_2 is:

$$u_2(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[A_{nm} \cos\left(k_{nm} t\right) + B_{nm} \sin\left(k_{nm} t\right)\right] \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{L}\right),$$
$$k_{nm} = \sqrt{\frac{(n^2 + m^2)\pi^2}{L^2}}.$$

Consider a population which is seeded initially with a complex pattern of population sub-groupings of a large variety of spatial scales. Describe and contrast how the complexity of this pattern changes in time in both cases (this only needs to be a rough description).

Q3 Fickian no more

Consider the following model of a population of cells moving in a domain $[0, L]$:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^4 u}{\partial x^4} + f(u),$$

subject to the boundary conditions,

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = \frac{\partial^3 u}{\partial x^3}(0,t) = \frac{\partial^3 u}{\partial x^3}(L,t) = 0.$$

(a) Assume there is an equilibrium $u_0 > 0$ such that $f(u_0) = 0$. Linearize the model and determine necessary conditions on a, b, and f such that, for some $L > 0$, the system exhibits a Turing instability.

(b) Give a biological interpretation of the three terms on the right-hand side of the equation (remember that u represents the density of a population). What do the boundary conditions signify physically? *Hint: Note carefully the signs of a and b in terms of constraints needed for Turing instability above.*

Q4 Bringing the budworms back

Consider the following modified spruce budworm model:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) - \frac{u^2}{1+u}, \quad x \in [0, L],$$

with homogeneous Dirichlet ($u = 0$) boundary conditions at $x = 0$ and $x = L$.

(a) Find all feasible spatially homogeneous equilibria and determine their stability in the absence of diffusion.

(b) Using a linear stability analysis, determine a value L_c such that for $L > L_c$, the population will not go extinct.

SECTION B

Q5 It's predator–prey, but not as we know it...

Consider the following interacting population model:

$$\begin{aligned}\frac{\mathrm{d}\widehat{u}}{\mathrm{d}\widehat{t}} &= a\widehat{u}(1-\widehat{u}) + c\widehat{u}\,\widehat{v}(1-\widehat{v}),\\ \frac{\mathrm{d}\widehat{v}}{\mathrm{d}\widehat{t}} &= -b\widehat{v} + d\widehat{u}\,\widehat{v},\end{aligned} \tag{1}$$

where $a, b, c, d > 0$ are real constants.

(a) Describe the physical interpretation of the terms on the right-hand side of the model.

(b) With a suitable rescaling, (1) can be written as:

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= u\,(1-\alpha u) + uv\,(1-\beta v)\,,\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -v\gamma + uv.\end{aligned}$$

State the form of the constants α, β, γ in terms of the constants a, b, c, d.

(c) Find all the physically valid equilibria, and determine conditions for them to be asymptotically stable.

Q6 Tidal advection–diffusion

Consider a species of algae represented by its density (number per unit length) $u(x,t)$ living in a thin pipe of length L which is fed into by the ocean at one end. The bacteria move subject to diffusion within the sea water, determined by Fick's law. They are also moved by the tides which we assume wax and wane (or advance and retreat) at a rate $v\sin(\omega t)$ (v and ω constant).

(a) Explain why the bacteria could be modelled using the following PDE based on the information given above:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + v\sin(\omega t)\frac{\partial u}{\partial x}. \tag{2}$$

You should start from the general advection–diffusion equation:

$$\frac{\partial u}{\partial t} = -\nabla \cdot \boldsymbol{J} + \nabla \cdot (\boldsymbol{v}u) + f,$$

where the terms in the equation are as defined in class.

(b) Assume v is sufficiently small such that (2) can be approximated as:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2},$$

and assume further that the algae density has the same value u_c at both ends of the pipe. Use separation of variables to show that

$$u(x,t) = \sum_{n=1}^{\infty} C_n \exp\left(-D\frac{n^2\pi^2}{L^2}t\right)\sin\left(\frac{n\pi x}{L}\right) + u_c, \tag{3}$$

for some real constants C and λ. You may quote any results derived in class.

(c) The previous solution indicates that in the absence of the tidal velocity v the population tends to a constant value over time. We now investigate whether the tidal flow can induce the growth of patterns.

Assume $v = \varepsilon \ll 1$. We consider a solution to (2) in the form $u(x,t) = u_0(x,t) + \varepsilon u_1(x,t)$ where $u_0(x,t)$ is (3). You should assume the solution u_1 satisfies **no-flux** boundary conditions (so the advective motion can change the average density). By seeking the general form for a solution for u_1 evaluate whether patterns can develop in the long time limit. Hint: You do not **need** to provide the full solution to answer this question, just clarify its form.

You may use any results derived in class.

Q7 Golden spiralling down the rabbit hole

Recall Fibonacci's model of a growing rabbit population,

$$F_n = F_{n-1} + F_{n-2}.$$

Remember that this model represents a month of time per generation, and that it takes one generation for the population to mature to breeding age.

(a) Introduce the variable $I_n = F_{n-1}$ to model the population of immature rabbits who cannot yet mate. Use this to write the system as a first-order system in terms of the mature rabbits F_n and the immature ones I_n. Solve this linear system to find a formula for the total number of mature rabbits at each generation, that is F_n. *You do not need to compute the eigenvectors or use the initial conditions, so your formula for F_n can contain arbitrary constants.*

(b) Consider a model of the form

$$u_n = u_{n-1}(1 - u_{n-1}) + v_{n-1}(1 - v_{n-1}), \quad v_n = u_{n-1}, \tag{4}$$

where now u_n and v_n represent densities of rabbits, rather than numbers of individuals. Besides treating rabbits in terms of densities, what else is different about this model compared with the one you wrote in part (a)? What is a plausible biological interpretation for any terms which are different?

(c) Find the equilibria of (4) and determine their stability.

Q8 Hunting with gradients

Consider the following reaction–cross-diffusion system,

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= u(K-u) - uv + \nabla^2 u + \alpha \nabla \cdot (u \nabla v)\,, \\
\frac{\partial v}{\partial t} &= -v + uv + \nabla^2 v + \beta \nabla \cdot (v \nabla u)\,,
\end{aligned}
\tag{5}
$$

where $K > 0$ on some closed and bounded domain Ω. Assume that the populations u and v are not allowed to leave this domain through the boundary, $\partial\Omega$.

(a) Give an interpretation to the uv term in the kinetics, and hence describe how the two populations interact.

(b) State what boundary conditions represent the populations not leaving through the boundary. Show that these conditions are equivalent to imposing Neumann boundary conditions.

(c) Compute the homogeneous equilibrium where both species coexist, and determine any conditions necessary for its feasibility and stability. Can this system satisfy the Turing conditions if we only have reaction and diffusion (that is, if $\alpha = \beta = 0$)?

(d) In the lecture notes we derived the following two necessary conditions for diffusion-driven (Turing) instability in cross-diffusion systems:

$$d_1 G_v + d_4 F_u > d_2 G_u + d_3 F_v,$$

$$(d_1 G_v + d_4 F_u - d_2 G_u - d_3 F_v)^2 > 4(d_1 d_4 - d_2 d_3)(F_u G_v - F_v G_u).$$

Write the functions d_i for the system (5). Consider $K = 2$, and consider two separate cases: (i) $\alpha = 0$, $\beta \neq 0$ and (ii) $\beta = 0$, $\alpha \neq 0$. For each case, determine if the system can admit Turing instabilities, and if so find a simple bound for α or β in each case which are necessary for such instabilities.

(e) Given the bounds on α and β found above, give a physical interpretation to the two cross-diffusion terms in the case that the system can admit Turing instabilities. Are these interpretations biologically plausible?