EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH3101-WE01

Title:
Fluid Mechanics III

Time:	3 hours	
Additional Material provided:	Formula sheet.	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 A fluid moves two-dimensionally so that its velocity, $\boldsymbol{u}$, is given by

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{1+t}\,\widehat{\boldsymbol{e}}_x + \widehat{\boldsymbol{e}}_y,$$

where $\widehat{\boldsymbol{e}}_x$ and $\widehat{\boldsymbol{e}}_y$ are the unit vectors for the Cartesian coordinates (x, y).

(a) Obtain equations in terms of x and y alone for:

(i) the streamline through $(-1, 1)$ at time $t = 0$,

(ii) the particle path for a particle released from $(-1, 1)$ at time $t = 0$,

(b) Write down an example of a velocity field where particle paths, streamlines and streaklines through a given point all coincide. (You do not have to show that this is true.)

Q2 (a) Write down expressions for the Cartesian components of a two-dimensional flow, $\boldsymbol{u} = (u, v)$, in terms of a velocity potential, ϕ, and a stream function, ψ. Under what conditions do ϕ and ψ exist?

(b) A line vortex with circulation Γ is placed in a two-dimensional potential flow which is otherwise stationary. The line vortex is placed at the coordinate $(0, 1)$ and there is a wall along the x-axis. The line vortex is seen to move parallel to the wall. Work out its velocity in the x-direction.

Hint: You may wish to use the result

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}(x)\right] = \frac{1}{x^2+1}.$$

Q3 Consider the inviscid Burgers equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

(a) What is the relevance of this equation to fluid mechanics?

(b) For each of the following initial conditions, $u(x, 0) = u_0(x)$, find an equation for the characteristics and confirm whether the inviscid Burgers equation has a smooth solution for all $t > 0$. If not, find the earliest time at which such a solution ceases to exist.

(i) $u_0(x) = 1$,

(ii) $u_0(x) = \sin(x)$.

Q4 (a) By nondimensionalising the incompressible unforced Navier–Stokes equations appropriately, show that in the limit of low Reynolds number (which you should define), the system reduces to the Stokes equations,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0, \\ \nabla^2 \boldsymbol{u} &= \boldsymbol{\nabla} p. \end{aligned}$$

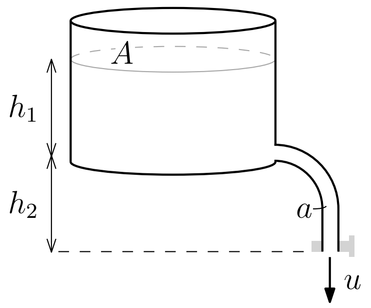
(b) Suppose $\boldsymbol{u}_1$ and p_1 are solutions to the Stokes equations in a domain with the boundary condition $\boldsymbol{u}_S(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})$ on the boundary S, for some function $\boldsymbol{g}$.

(i) If the boundary condition is changed to $\boldsymbol{u}_S(\boldsymbol{x}) = -\boldsymbol{g}(\boldsymbol{x})$, find a new solution to the Stokes equations which implies that Stokes flow is reversible.

(ii) What can we therefore conclude about the swimming strokes of micro-organisms in Stokes flow?

SECTION B

Q5 In a small pub, beer is stored in a cask which can be modelled as a tank with uniform horizontal cross-sectional area A. The beer is poured from the tank by opening a tap which is connected to the bottom of the cask through a tube with cross-sectional area $a < A$. The height of the beer in the cask is h_1, and the tap is placed a distance h_2 below the bottom of the cask, like so:



We will model the beer as an incompressible, inviscid fluid, flowing steadily under gravity.

(a) Starting with the Euler equations, show that

$$\frac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho_0} + \Phi$$

is constant along any streamline, where Φ is a scalar function you should define.

(b) Hence show that the speed of the beer leaving the tap, u, satisfies

$$u^2 = \left(1 - \frac{a^2}{A^2}\right)^{-1} 2gf(h_1, h_2),$$

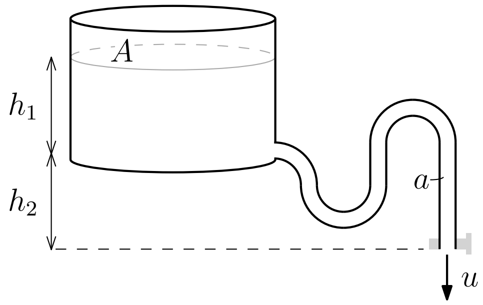
where you should state the function $f(h_1, h_2)$.

(c) The pub landlady wants to increase the speed at which glasses are filled from the end of the tap. Three regular customers – Anthony, Luci and Chris – suggest three different options. *Briefly*, comment on the effectiveness of each of these suggestions.

(i) Anthony: 'Increase a.'

(ii) Luci: 'Place everything in a lift which is accelerating upwards.'

(iii) Chris: 'Change the shape of the pipe so it looks like this:'



Q6 The equations governing the dynamics of water waves in the cylindrical domain

$$\{(r,\theta,z) : 0 < r < a,\ -h < z < \eta(r,\theta,t)\}$$

are

$$\nabla^2\phi = 0 \qquad \text{in } -h < z < \eta, \tag{1}$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad \text{on } z = -h, \tag{2}$$

$$\frac{\partial\phi}{\partial r} = 0 \qquad \text{on } r = a, \tag{3}$$

$$\frac{\partial\eta}{\partial t} + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\eta = \frac{\partial\phi}{\partial z} \qquad \text{on } z = \eta, \tag{4}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g\eta = 0 \qquad \text{on } z = \eta. \tag{5}$$

(a) Provide a one-sentence physical interpretation of each of equations (1) to (5).

(b) Assuming small amplitude water waves, carefully linearise the system of equations by introducing a parameter $\varepsilon \ll 1$, stating any assumptions you make.

(c) Consider a solution of the form

$$\phi(r,z,t) = R(r)Z(z)\mathrm{e}^{\mathrm{i}\omega t}$$

for the linearised system. Derive the dispersion relation and show it can be expressed as

$$\omega^2 = \frac{g}{h}f(kh),$$

for a function f you should specify, and where k is a constant.

(d) By solving for $R(r)$, show that there are a discrete number of permitted frequencies, ω, and that their dependence on k in the dispersion relation are such that $x = ka$ is a turning point of the Bessel function $J_0(x)$.

Q7 A corridor of length L and width a is modelled in 2D as the domain

$$\{(x, y) : 0 < x < L,\, 0 < y < a\}.$$

At $x = 0$, the corridor has doors which are sealed closed; whereas at $x = L$, the corridor has doors which are open. We consider linear sound waves inside the corridor, described by $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ where

$$\frac{\partial^2 \phi}{\partial t^2} = c_0 \nabla^2 \phi.$$

(a) Write down the appropriate boundary conditions for ϕ on all four boundaries.

(b) By assuming a solution of the form

$$\phi(x, y, t) = X(x)Y(y)\mathrm{e}^{-\mathrm{i}\omega t},$$

show that X and Y satisfy the equations

$$X'' = -k_x^2 X, \quad Y'' = -k_y^2 Y,$$

and find k_y as a function of k_x.

(c) Hence find the possible values of ω for these waves.

(d) The closed doors are now opened. How does this change the possible values of ω?

(e) Your friend is shouting at you from the opposite end of the corridor to where you are standing, but it is hard to understand what they are saying. By considering the phase speed in the x-direction, explain why it is legitimate to blame the physics of the problem for your lack of comprehension.

Q8 Incompressible viscous fluid of kinematic viscosity ν and density ρ_0 is in a channel formed by two parallel plates at $y = 0$ and $y = h$. The lower wall, at $y = 0$, is at rest whereas the upper wall moves in the positive x-direction with constant speed U. A constant pressure gradient is maintained in the flow,

$$\frac{\partial p}{\partial x} = -\rho_0 G,$$

for a constant G. You may assume that gravitational effects are negligible.

(a) Starting with the unforced incompressible Navier–Stokes equations, show that a steady flow in the channel of the form

$$\boldsymbol{u} = u(y)\widehat{\boldsymbol{e}}_x$$

is possible, with the velocity profile $u(y)$ given by a linear combination of the Poiseuille and Couette flow profiles.

(b) Given positive G, and assuming fixed values of h, ν, ρ_0 and U, the flow always moves in the positive x-direction. Show that, however, for a sufficiently large negative value of G, which you should find, the flow near the lower wall instead moves in the negative x-direction. Determine the thickness of this reversed-flow layer.

(c) Sketch the typical velocity profiles in the flow for various combinations of the flow parameters, including profiles with and without reversed flow.

(d) What value of G is needed so that the net flow through the channel is zero?