EXAMINATION PAPER

Examination Session:	**Year:**	**Exam Code:**
May/June	2025	MATH3111-WE01

Title:
Quantum Mechanics III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 (a) Write the definition of a unitary operator $\hat{U}$.

(b) Show that an operator $\hat{U}$ is unitary if and only if it preserves the inner product between any two states.

(c) If $\hat{U}$ and $\hat{V}$ are two unitary operators, show that the product $\hat{U}\hat{V}$ is also unitary.

Q2 The Hilbert space of a two-dimensional Quantum Mechanical system has an orthonormal basis $B = \{|a\rangle,\ |b\rangle\}$. Consider the observable operator $\hat{M}$, where

$$\hat{M} = \alpha\,|a\rangle\langle a| + \beta|b\rangle\langle a| + 2i|a\rangle\langle b|\,,$$

and where α, β are constants.

(a) How is α constrained, and what is the value of β?

(b) At time $t = 0$ a measurement of $\hat{M}$ is made and a value of -1 found. Use this information to completely determine the matrix form of $\hat{M}$ in the basis B.

(c) What is the state $|\psi\rangle$ immediately after the measurement of $\hat{M} = -1$?

(d) What is the other possible outcome of a measurement of $\hat{M}$ that could have been found?

(e) Suppose the state evolves under the influence of some Hamiltonian $\hat{H}$. What property of $\hat{H}$ is required for there to be a non-zero probability of measuring the other possible value of $\hat{M}$ at a later time?

Q3 State what are the angular momentum operators $J_i, (i = x, y, z)$ in three dimensions and state their basic commutation relations, including those with the total angular momentum operator $\hat{J}^2$.

(a) Using the basic commutation relations evaluate the following commutators:

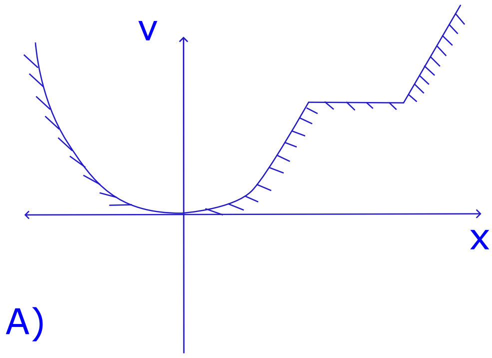
$$[\hat{J}^2, \hat{J}_z^2] \qquad [\hat{J}_z, \hat{x}\hat{p}_y - \hat{y}\hat{p}_x]$$

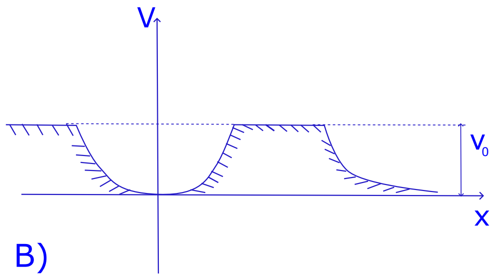
(b) Evaluate the following commutator explicitly

$$[\hat{J}_z, \hat{x}^2 + \hat{y}^2]\,,$$

and explain the physical significance of the result in terms of rotational invariance.

Q4 A quantum particle with energy E is propagating in the potentials A) and B) shown in the images below. For potential A), as $x \to \pm\infty, V(x) \to \infty$. For potential B), as $x \to -\infty, V(x) \to V_0$, and as $x \to \infty$, $V(x) \to 0$. Based on the general properties of quantum systems, answer the following questions:





(a) What is the nature of the quantum spectrum of the particle in these two potentials? Is it continuous or discrete?

(b) Sketch the qualitative form of the wave function for both systems, indicating where the particle is most likely to be found.

SECTION B

Q5 A charged particle in an external magnetic field $\boldsymbol{B}$ has a Hamiltonian that includes the terms

$$\hat{H} \;=\; -\frac{e}{2m}\hat{\boldsymbol{L}}\cdot\boldsymbol{B}$$

where the angular momentum in the "natural" representation is given by the following system of matrices:

$$\hat{L}_x \;=\; i\hbar\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix} \;\;;\;\; \hat{L}_y \;=\; i\hbar\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix} \;\;;\;\; \hat{L}_z \;=\; i\hbar\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\;.$$

Assume that the magnetic field points in the z direction, $\boldsymbol{B} = (0, 0, B_z)$. At time $t = 0$, the angular momentum is measured in the z direction to have value $L_z = \hbar$.

(a) What is the wavefunction of the state immediately after this measurement?

(b) Find the state of the system at any later time, t.

(c) Hence find $\langle \hat{\boldsymbol{L}}(t)\rangle$

(d) Determine $\langle \hat{\boldsymbol{L}}(t)^2\rangle = \langle \hat{L}_x^2\rangle + \langle \hat{L}_y^2\rangle + \langle \hat{L}_z^2\rangle$.

(e) Demonstrate that your result is consistent with Ehrenfest's theorem, namely that

$$\frac{d\langle \hat{\boldsymbol{L}}\rangle}{dt} = \frac{-i}{\hbar}\langle [\hat{\boldsymbol{L}}, H]\rangle.$$

Q6 The Hamiltonian for the simple harmonic oscillator is given by

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \ .$$

In addition we define the annihilation and creation operators as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}\left(i\hat{p} + m\omega\hat{x}\right) \ , \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(-i\hat{p} + m\omega\hat{x}\right) \ .$$

(a) Using the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ determine the commutator $[\hat{a}, \hat{a}^\dagger]$.

(b) Use your result to express $\hat{H}$ in terms of the number operator, $\hat{n} = \hat{a}^\dagger\hat{a}$.

(c) Hence show that

$$[\hat{a}, \hat{H}] \ = \ \omega\hat{a} \ \ ; \quad [\hat{a}^\dagger, \hat{H}] \ = \ -\omega\hat{a}^\dagger \ . \tag{1}$$

(d) A "squeezed state" can be thought of as a ground state of the 'wrong' harmonic oscillator. That is, instead of obeying the usual condition for a ground state that $\hat{a}|\psi\rangle = 0$ or equivalently $(m\omega\hat{x} + i\hat{p})|\psi\rangle = 0$, a squeezed state satisfies

$$(m\,\Omega(t)\,\hat{x} + i\hat{p})\,|\psi\rangle = 0 \tag{2}$$

where $\Omega(t) \neq \omega$ is generally complex and a function of time. Show that

$$\langle\, x \,|\, \psi\,\rangle \ = \ \left(\frac{m\mathrm{Re}(\Omega)}{\pi\hbar}\right)^{1/4} e^{-m\Omega x^2/2\hbar}$$

is a properly normalised squeezed state.

(e) Such a squeezed state must evolve according to Schrödinger's equation, namely as $|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle$. If we define the initial value of Ω so that it is real, $\Omega(0) = \omega_0$, use this fact and the relations in Eqs. (1) and (2) to find $\Omega(t)$ in terms of ω_0 and ω.

(f) Hence show that the width of the squeezed state, $\Delta x = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2}$, oscillates in time as

$$\Delta x(t) \ = \ \Delta x(0)\sqrt{1 + \frac{\omega_0^2 - \omega^2}{\omega^2}\sin^2\omega t}$$

where $\Delta x(0)$ is its value at $t = 0$. (**Hint**: You may use $\int_{-\infty}^{\infty} e^{-ax^2}x^2 = \frac{\sqrt{\pi}}{2a^{3/2}}$.)

Q7 A quantum particle is in a 3D spherically symmetric potential $V(r) = r^4$. If the Laplacian in 3D is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{\hbar^2 r^2},$$

where the angular momentum operator satisfies

$$\hat{L}^2 Y(\theta, \phi) = l(l+1)\hbar^2 Y(\theta, \phi),$$

answer the following questions.

(a) Starting from the time-independent Schrödinger equation, derive the angular and radial equations for this system. Outline all the steps when deriving the equations.

(b) Solve the angular part of the Schrödinger equation and write the general form of the angular wavefunction. Based on the angular solution, determine whether the energy spectrum of the system is degenerate. Briefly explain your reasoning.

(c) Write the effective potential $V_{\text{eff}}(r)$ for the radial equation, including the centrifugal term. Sketch $V_{\text{eff}}(r)$ for $l = 0$ and $l \neq 0$. Comment on the qualitative features of the potential.

(d) Focus on the spherically symmetric case:

(i) Using the WKB approximation, derive the quantization condition for the energy levels of the system. Show explicitly that $E_n \sim (n + 1/2)^{4/3}$.

(ii) Discuss whether the energy levels are equidistant. Please, compare this behavior to the harmonic oscillator ($V(r) = r^2$) and explain how the spacing changes with n increasing.

Q8 A quantum particle is propagating in a 2D simple harmonic oscillator potential. The unperturbed Hamiltonian for a unit mass is given by

$$H_0 = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{1}{2}\omega^2(x^2 + y^2).$$

A perturbation is introduced in the system as

$$\delta V = a(x^2 - y^2) + b\,,$$

where a and b are positive numbers.

If you know that $\hat{x}$ and $\hat{y}$ expressed in term of creation and annhilation operators are

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega}}\left(\hat{a}_x + \hat{a}_x^\dagger\right), \quad \hat{y} = \sqrt{\frac{\hbar}{2\omega}}\left(\hat{a}_y + \hat{a}_y^\dagger\right)\,, \tag{3}$$

answer the following questions:

(a) Write the energy spectrum of the unperturbed system. Explain whether the energy levels are degenerate and determine the degeneracy of the n-th energy level.

(b) Express δV in terms of the ladder operators $\hat{a}_x, \hat{a}_x^\dagger, \hat{a}_y, \hat{a}_y^\dagger$ and work out the first-order correction to the ground-state energy as well to the state, due to the perturbation.

(c) Apply degenerate perturbation theory to evaluate the energy shifts for the first excited state ($n = 1$). Explain the physical significance of the perturbation δV and how it affects the symmetry and degeneracy of the unperturbed system.

(d) Work out the approximate energy of the states (n_x, n_y), $n_x, n_y \gg 1$, due to the perturbation.(**Hint:** In this part it is useful to look at *structure* of the matrix δV as well how different elements in the matrix depend on n_x, n_y, and do that for some explicit examples, for eg $n = 3$ or $n = 5$.)