Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH3141-WE01

Title:
Operations Research III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Mr Monkey has a daily workout routine, which consists of either running, push-ups, or yoga. Each day, he picks his routine based on the routine from the previous day, and he never picks the same routine on consecutive days.

- After running, he will pick push-ups or yoga with equal probability.
- After push-ups, he is three times as likely to run as opposed to doing yoga.
- After yoga, he always picks running.

Per daily workout, running burns 500 calories, push-ups burn 350 calories, and yoga burns 180 calories. Find the long-run expected daily calories burned by Mr Monkey.

Q2 Miss Butterfly is hungry, and can drink up to 9 millilitres of nectar in the next few hours. Her flower field grows butterflyweed, coneflower, and zinnia. Each flower gives a certain amount of nectar, as well as a reward reflecting her preferences:

flower	nectar per flower (millilitres)	reward per flower
butterflyweed	1	2
coneflower	3	4
zinnia	2	5

When visiting a flower, she must deplete all available nectar. To maintain a healthy diet, she must drink at least 2 millilitres of butterflyweed nectar, and no more than 4 millilitres of zinnia nectar.

Use dynamic programming to find the selection of flowers that maximizes Miss Butterfly's total reward while maintaining a healthy diet.

Q3 Solve the following linear programming problem using the two-phase method, and find all feasible values of x_1, x_2 and x_3 for which the optimum value is attained.

$$\begin{array}{llllllll} \max & 2x_1 & - & x_2 & + & 4x_3 & & \\ \text{subject to} & -x_1 & - & 2x_2 & + & x_3 & = & 2 \\ & -x_1 & + & 2x_2 & + & 2x_3 & \leq & 5 \\ & x_1 & & & & & \geq & 0 \\ & & & x_2 & & & \geq & 0 \\ & & & & & x_3 & \geq & 0. \end{array}$$

Q4 Starting from the north-west initial assignment, find all optimal basic feasible solutions for the transportation problem with costs, supplies and demands given below:

$$[c_{ij}] = \begin{bmatrix} 4 & 2 & 1 & 4 \\ 6 & 1 & 3 & 1 \\ 6 & 3 & 4 & 2 \end{bmatrix}, \qquad [a_i] = \begin{bmatrix} 10 \\ 40 \\ 35 \end{bmatrix}, \qquad [b_j] = \begin{bmatrix} 20 & 25 & 30 & 10 \end{bmatrix}.$$

SECTION B

Q5 Miss Giraffe participates in a quiz show that has three types of questions. Each question type has a value, and a probability of Miss Giraffe correctly answering it:

type	value	probability of correct answer
1	10	0.8
2	30	0.6
3	70	0.5

After a correctly answered type i question ($i < 3$), Miss Giraffe can either

- bank the value of that question in which case the next question is of type 1, or
- receive the next question in which case she banks nothing but the next question is of type $i + 1$.

If Miss Giraffe correctly answers a type 3 question, she automatically banks the value and returns to type 1. If she answers incorrectly, she banks nothing and returns to type 1.

Use policy improvement to find a policy that maximises Miss Giraffe's long-run amount banked per question asked, starting with Miss Giraffe's current policy of always banking as soon as possible (i.e. never accepting questions of any type other than type 1). Hint: As state, choose the question type that is *about to be* asked.

Q6 (a) State and prove the one-step equations for the total expected discounted reward of a Markov decision process under a *non-stationary* policy.

(b) Suggest and explain an algorithm based on these equations to calculate (or, approximate to any desired accuracy) the total expected discounted reward under a *non-stationary* policy.

Q7 A linear programming problem in the form $\max c^{\mathsf{T}}x$ subject to $Ax \leq b$ and $x \geq 0$ is solved using the simplex algorithm and the final table is

T_*	x_1	x_2	x_3	s_1	s_2	s_3	
z	0	0	1	3	2	0	235
x_2	0	1	-1	-1	2	0	15
s_3	0	0	2	-1	1	1	10
x_1	1	0	5	1	-1	0	20

Use post-optimal analysis to answer all of the following questions:

(a) Find the optimal solution and the value of the objective function when the original parameters $b = (b_1, b_2, b_3)^{\mathsf{T}}$ are modified by decreasing b_1 by 10 and increasing b_2 by 5.

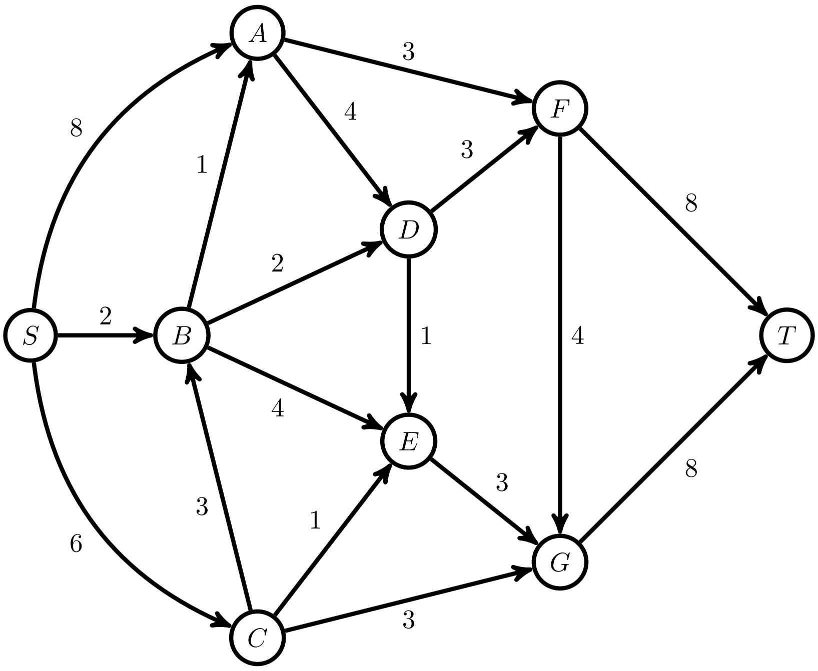
(b) With the original parameter values for b, suppose that $c^{\mathsf{T}} = (c_1, c_2, c_3)$ is modified by increasing c_3 by 2. Show that this makes the current basis non-optimal and pivot to find the optimal solution.

(c) With the original parameter values for b and c suppose now that a new constraint

$$x_1 + 2x_2 + x_3 \leq 46$$

is added. Calculate the new table formed from T_* by adding a row for the new constraint, adding a column for the new slack variable s_4 and making s_4 basic. Is the basis $\{x_2, s_3, x_1, s_4\}$ feasible?

Q8 Consider a transportation system that can be represented by the following flow network, with capacities as labelled:



(a) Apply the Ford–Fulkerson algorithm to find the maximum flow through the network from source node S to terminus node T.

(b) Planners are considering improvements to the transportation system, with the possibility to increase the capacity along a single arc in the network. Identify all arcs that provide an opportunity for increasing the maximum flow, and provide a theoretical argument as to why only these arcs provide this opportunity. You may use, without proof, any theorems from the course, as long as you state them clearly.

(c) What is the optimal maximum flow that can be achieved by such an improvement (i.e., with an unlimited increase in the capacity along a single arc)?