

EXAMINATION PAPER

Examination Session: May/June

2025

Year:

Exam Code:

MATH3231-WE01

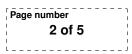
Title:

Solitons III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Answer all questions. Section A is worth 40% and Section B is worth 60%. Within
each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with
barcodes.
Begin your answer to each question on a new page.

Revision:



SECTION A

- Q1 Write down the ball and box rule for one time-step $t \to t + 1$ in the single-colour ball and box model. If at time t = 0 there are balls in boxes 1, 2, 3 and 8, find the locations of the balls at times t = 1, t = 2 and t = 3, and give the phase shifts undergone by the two solitons during this process.
- Q2 Consider the pair of equations

$$u_x + v_y = 0$$
$$u_y + 2v_x = 0$$

which form a Bäcklund transformation relating an equation for u(x, y) to one for v(x, y).

- (a) Cross-differentiating, find the equations satisfied by u and v.
- (b) Check that v(x, y) = x is a solution to your equation for v, and find the corresponding solution to your equation for u.
- Q3 (a) Solve the Marchenko equation

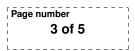
$$K(x,z) + F(x+z) + \int_{-\infty}^{x} dy \, K(x,y) \, F(y+z) = 0$$

to determine the unknown function K(x, z), given that $F(x) = c \exp(cx)\theta(x)$, where c is a constant and

$$\theta(x) = \begin{cases} 1 & x \ge 0\\ 0 & x < 0 \end{cases}.$$

(b) Calculate

$$V(x) = 2\frac{d}{dx}K(x,x) \ .$$



Q4 The Poisson bracket (at time t) of two functionals F[u], G[u] of u(x, t) is defined as

Exam code

MATH3231-WE01

$$\{F[u], G[u]\} := \int_{-\infty}^{+\infty} dx \ \frac{\delta F[u]}{\delta u(x,t)} \frac{\partial}{\partial x} \frac{\delta G[u]}{\delta u(x,t)} \ ,$$

and the time evolution of a functional F[u] is governed by the equation

$$\frac{d}{dt}F[u] = \{H[u], F[u]\}$$

where H[u] is the Hamiltonian. In the following you may assume that fields fall off sufficiently fast as $x \to \pm \infty$ that all boundary terms vanish.

(a) Derive an expression for $\frac{\partial}{\partial t}u(x,t)$, by viewing u(x,t) as

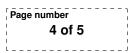
$$u(x,t) = \int_{-\infty}^{+\infty} dy \ \delta(y-x)u(y,t) \ .$$

(b) Find the PDE that governs the time evolution of u(x,t) if

$$H[u] = \int_{-\infty}^{+\infty} dx \left[u(x,t)^3 - \frac{1}{2} u_x(x,t)^2 \right] .$$

(c) Use the Poisson bracket and the Hamiltonian H[u] given in the previous part to calculate the time derivative of

$$Q[u] = \int_{-\infty}^{+\infty} dx \ u(x,t)^2$$



Exam code
MATH3231-WE01
1
L

SECTION B

Q5 (a) Suppose that the equation of motion for a field u(x,t) is such that

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

for a pair of densities $\rho(u, u_t, u_x, ...)$ and $j(u, u_t, u_x, ...)$. Show that the charge $Q = \int_{-\infty}^{\infty} \rho \, dx$ is conserved, provided the boundary conditions for u imply a certain condition on the limits of j which you should state.

(b) Find two independent conserved charges for the equation

$$u_t + 20u^3u_x + u_{xxx} = 0$$

with boundary conditions $u, u_x, u_{xx} \to 0$ as $x \to \pm \infty$.

Q6 A field u(x,t), defined on the infinite line $-\infty < x < \infty$, has energy

$$E[u] = \int_{-\infty}^{\infty} \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}u^4(u^2 - 1)^2 \, dx \, .$$

(a) If u is to have finite energy, what values can the topological charge

$$Q_0[u] = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} u \, dx = [u]_{x=-\infty}^{x=+\infty}$$

take? For each case, give the limits of u at $x = \pm \infty$ which lead to that charge.

- (b) Now suppose $Q_0[u] = Q^{(i)}$, where $Q^{(i)}$ is one of the possible values of the topological charge that you identified in part (a), and suppose in addition that $Q^{(i)} > 0$. Show that $E[u] \ge K^{(i)}$, where $K^{(i)}$ is a constant which you should find for each $Q^{(i)} > 0$.
- (c) Write down the conditions for the bounds you found in part (b) to be saturated, so that $E[u] = K^{(i)}$. Is it possible for these to be satisfied if $Q^{(i)}$ is the largest of the values that you found in part (a)? (You can assume that solutions *do* exist which saturate the bound for all other possible positive values of $Q^{(i)}$.)

Exam code MATH3231-WE01

Q7 The time-independent Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x)$$

for the potential $V(x) = -2 \operatorname{sech}^2(x)$ has the general solution

$$\psi(x) = Ae^{ikx} \left(-ik + \tanh(x)\right) + Be^{-ikx} \left(ik + \tanh(x)\right) ,$$

where A and B are constants and $k^2 > 0$.

(a) Find the reflection and transmission coefficients R(k) and T(k) for the potential

$$V(x) = \begin{cases} 0 , & x < 0 \\ -2 \operatorname{sech}^2(x) , & x \ge 0 . \end{cases}$$

(b) Find all (unnormalised) bound state solutions for the potential in part (a).

Q8 The pair of operators

$$L = (\phi_1 + i\phi_2) - 2iz \ \phi_3 + z^2(\phi_1 - i\phi_2)$$
$$M = i\phi_3 - z \ (\phi_1 - i\phi_2)$$

satisfies the Lax equation $\dot{L} = [M, L]$ for all values of the parameter z, where the dot denotes a time derivative, and ϕ_1, ϕ_2, ϕ_3 do not depend on z.

- (a) Find time evolution equations for ϕ_1, ϕ_2, ϕ_3 .
- (b) Assume that $\phi_n = i w_n \sigma_n / 2$ (indices n = 1, 2, 3 are not summed over), where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Find time evolution equations for the functions w_1, w_2, w_3 , and conservation laws for that time evolution.