

EXAMINATION PAPER

Examination Session: May/June

2025

Year:

Exam Code:

MATH3251-WE01

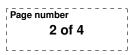
Title:

Stochastic Processes III

| Time: | 3 hours | |
|-------------------------------|---------|---|
| Additional Material provided: | | |
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| Matariala Darmittadı | | |
| Materials Permitted: | | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators |
| | | is forbidden. |

| Instructions to Candidates: | Answer all questions. | |
|-----------------------------|--|--|
| | Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. | |
| | Write your answer in the white-covered answer booklet with barcodes. | |
| | Begin your answer to each question on a new page. | |
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Revision:



SECTION A

- Q1 Let X, Y be two non-negative random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[X], \mathbb{E}[Y] < \infty$.
 - (a) State the definition of $X \leq_{st} Y$.
 - (b) Suppose $X \leq_{\text{st}} Y$. Show that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.
 - (c) Does $\mathbb{E}[X] \leq \mathbb{E}[Y]$ imply $X \leq_{st} Y$? Provide a proof if this is true, or a counter-example otherwise.
- **Q2** Let $(N(t))_{t>0}$ be a Poisson process with intensity λ . Assume that N(0) = 0.
 - (a) Write down a formula for the distribution of N(t) for fixed t > 0, that is, what is the value of $\mathbb{P}[N(t) = k]$ for integers $k \ge 0$?
 - (b) Let s, t > 0 with 0 < s < t. Explain why for m > n

$$\mathbb{P}[N(s) = m \,|\, N(t) = n] = 0,$$

and for $0 \le m \le n$ derive a formula for

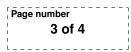
$$\mathbb{P}[N(s) = m \,|\, N(t) = n].$$

- Q3 (a) State the definition of a martingale sequence. Make sure to state all probabilistic objects and conditions involved in the definition.
 - (b) Let X_1, X_2, \ldots be a sequence of independent random variables with

$$\mathbb{P}[X_n = i] = \begin{cases} e^{-n} & \text{if } i = 0\\ 1 - 2e^{-n} & \text{if } i = 1\\ e^{-n} & \text{if } i = 2 \end{cases} \quad \text{for } n \ge 1.$$

Let $M_n = \prod_{i=1}^n X_i$. Specify a filtration and show that M_n is a martingale with respect to this filtration.

- **Q4** Let $(Z_n)_{n\geq 0}$ be a branching process with $Z_0 = 1$ and suppose $\varphi_n(s) := \mathbb{E}[s^{Z_n}]$ satisfies $\varphi_1(s) := \mathbb{E}[s^{Z_1}] = \frac{1}{5}(1+s+2s^2)$.
 - (a) Find $\mathbb{E}[Z_n]$ and $\mathbb{E}[Z_n^2]$ in terms of n.
 - (b) Find $Cov(Z_{2024}, Z_{2025})$.



SECTION B

- **Q5** (a) State the definition of the total variation distance $d_{\text{TV}}(U, V)$ between two real-valued random variables U, V.
 - (b) State and prove the triangle inequality for total variation distance.
 - (c) What is the largest value that the total variation distance may take? Prove this best upper bound and show that this can be attained with suitable examples of random variables.
 - (d) Let $(X_n)_{n\geq 0}$ be a branching process with $X_0 = 1$ and $\varphi_X(s) := \mathbb{E}[s^{X_1}] = \frac{1}{4}(1+s+s^2+s^3)$. Find the extinction probability $\rho_X := \lim_{n\to\infty} \mathbb{P}(X_n = 0)$.
 - (e) Continuing from the previous part, let $(Y_n)_{n\geq 0}$ be another branching process with $Y_0 = 10$ and

$$\varphi_Y(s) := \mathbb{E}[s^{Y_2}|Y_1 = 1] = \frac{1}{4}(1+s^3) + \frac{1}{2}e^{\frac{1}{2}(s-1)}.$$

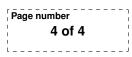
Evaluate $\lim_{n\to\infty} d_{\mathrm{TV}}(X_n, Y_n)$.

(Hint: what is the mean number of offspring of each individual in $(Y_n)_{n\geq 0}$?)

(You may use any results concerning branching processes provided that they are stated carefully.)

- **Q6** (a) What does it mean for a real-valued random variable X to be arithmetic? Give an example of a 3-arithmetic random variable X satisfying Var(X) = 2.
 - (b) Let d > 0. State the Blackwell renewal theorem for d-arithmetic distributions.
 - (c) You are generating a random sequence of alphabets in the following way:
 - Alphabets are picked independently according to the same distribution.
 - The distribution of each alphabet is as follows: with probability ¹/₄, choose a vowel {A, E, I, O, U} uniformly at random; otherwise (i.e. with probability ³/₄), choose one of the 21 consonants uniformly at random.

Let τ be the time for the first appearance of MEME in consecutive order. Find $\mathbb{E}[\tau]$ by applying results for suitable renewal processes carefully.



Q7 Let $X_1 = (X_1(t))_{t \ge 0}$ and $X_2 = (X_2(t))_{t \ge 0}$ be continuous time Markov processes on $\{0, 1\}$, with common generator

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} \qquad \alpha, \beta > 0.$$

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Assume that the processes X_1 and X_2 are independent, that is $X_1(u)$ and $X_2(v)$ are independent for all $u, v \ge 0$. Let $X(t) = X_1(t) + X_2(t)$ for all $t \ge 0$.

- (a) Show that $X = (X(t))_{t \ge 0}$ is a continuous time Markov process on $\{0, 1, 2\}$ and find its generator.
- (b) Show that X is an irreducible Markov process.
- (c) Find the stationary distribution of X.
- (d) Find

$$\lim_{t \to \infty} \mathbb{P}[X(t) = 1 \mid X(0) = 0].$$

Q8 At time n = 0, an urn contains 1 blue ball and 1 red ball. At each time n = 1, 2, 3, ..., a ball is chosen at random from the urn and returned to the urn together with a new ball of the same colour. Just after time n, there are n + 2 balls in the urn of which B_n are blue and R_n are red. Let

$$M_n = \frac{B_n}{B_n + R_n} = \frac{B_n}{n+2}$$

be the proportion of blue balls in the urn after time n.

- (a) Show that M_n is a martingale sequence. You should specify the filtration.
- (b) Show that for an integer k with $1 \le k \le n+1$,

$$\mathbb{P}[B_n = k] = \frac{1}{n+1}.$$

(c) Let T be the first time a blue ball is added to the urn. Compute

$$\mathbb{E}\left[\frac{1}{T+2}\right]$$

carefully stating any theorems you use from the course.