



EXAMINATION PAPER

Examination Session: May/June	Year: 2025	Exam Code: MATH3251-WE01
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Title: Stochastic Processes III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>	
		Revision:

SECTION A

Q1 Let X, Y be two non-negative random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[X], \mathbb{E}[Y] < \infty$.

- (a) State the definition of $X \leq_{\text{st}} Y$.
- (b) Suppose $X \leq_{\text{st}} Y$. Show that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.
- (c) Does $\mathbb{E}[X] \leq \mathbb{E}[Y]$ imply $X \leq_{\text{st}} Y$? Provide a proof if this is true, or a counter-example otherwise.

Q2 Let $(N(t))_{t \geq 0}$ be a Poisson process with intensity λ . Assume that $N(0) = 0$.

- (a) Write down a formula for the distribution of $N(t)$ for fixed $t > 0$, that is, what is the value of $\mathbb{P}[N(t) = k]$ for integers $k \geq 0$?
- (b) Let $s, t > 0$ with $0 < s < t$. Explain why for $m > n$

$$\mathbb{P}[N(s) = m \mid N(t) = n] = 0,$$

and for $0 \leq m \leq n$ derive a formula for

$$\mathbb{P}[N(s) = m \mid N(t) = n].$$

- Q3** (a) State the definition of a martingale sequence. Make sure to state all probabilistic objects and conditions involved in the definition.
- (b) Let X_1, X_2, \dots be a sequence of independent random variables with

$$\mathbb{P}[X_n = i] = \begin{cases} e^{-n} & \text{if } i = 0 \\ 1 - 2e^{-n} & \text{if } i = 1 \\ e^{-n} & \text{if } i = 2 \end{cases} \quad \text{for } n \geq 1.$$

Let $M_n = \prod_{i=1}^n X_i$. Specify a filtration and show that M_n is a martingale with respect to this filtration.

Q4 Let $(Z_n)_{n \geq 0}$ be a branching process with $Z_0 = 1$ and suppose $\varphi_n(s) := \mathbb{E}[s^{Z_n}]$ satisfies $\varphi_1(s) := \mathbb{E}[s^{Z_1}] = \frac{1}{5}(1 + s + 2s^2)$.

- (a) Find $\mathbb{E}[Z_n]$ and $\mathbb{E}[Z_n^2]$ in terms of n .
- (b) Find $\text{Cov}(Z_{2024}, Z_{2025})$.

SECTION B

- Q5** (a) State the definition of the total variation distance $d_{TV}(U, V)$ between two real-valued random variables U, V .
- (b) State and prove the triangle inequality for total variation distance.
- (c) What is the largest value that the total variation distance may take? Prove this best upper bound and show that this can be attained with suitable examples of random variables.
- (d) Let $(X_n)_{n \geq 0}$ be a branching process with $X_0 = 1$ and $\varphi_X(s) := \mathbb{E}[s^{X_1}] = \frac{1}{4}(1 + s + s^2 + s^3)$. Find the extinction probability $\rho_X := \lim_{n \rightarrow \infty} \mathbb{P}(X_n = 0)$.
- (e) Continuing from the previous part, let $(Y_n)_{n \geq 0}$ be another branching process with $Y_0 = 10$ and

$$\varphi_Y(s) := \mathbb{E}[s^{Y_1} | Y_0 = 1] = \frac{1}{4}(1 + s^3) + \frac{1}{2}e^{\frac{1}{2}(s-1)}.$$

Evaluate $\lim_{n \rightarrow \infty} d_{TV}(X_n, Y_n)$.

(Hint: what is the mean number of offspring of each individual in $(Y_n)_{n \geq 0}$?)

(You may use any results concerning branching processes provided that they are stated carefully.)

- Q6** (a) What does it mean for a real-valued random variable X to be arithmetic? Give an example of a 3-arithmetic random variable X satisfying $\text{Var}(X) = 2$.
- (b) Let $d > 0$. State the Blackwell renewal theorem for d -arithmetic distributions.
- (c) You are generating a random sequence of alphabets in the following way:
- Alphabets are picked independently according to the same distribution.
 - The distribution of each alphabet is as follows: with probability $\frac{1}{4}$, choose a vowel $\{A, E, I, O, U\}$ uniformly at random; otherwise (i.e. with probability $\frac{3}{4}$), choose one of the 21 consonants uniformly at random.

Let τ be the time for the first appearance of MEME in consecutive order. Find $\mathbb{E}[\tau]$ by applying results for suitable renewal processes carefully.

Q7 Let $X_1 = (X_1(t))_{t \geq 0}$ and $X_2 = (X_2(t))_{t \geq 0}$ be continuous time Markov processes on $\{0, 1\}$, with common generator

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} \quad \alpha, \beta > 0.$$

Assume that the processes X_1 and X_2 are independent, that is $X_1(u)$ and $X_2(v)$ are independent for all $u, v \geq 0$. Let $X(t) = X_1(t) + X_2(t)$ for all $t \geq 0$.

- (a) Show that $X = (X(t))_{t \geq 0}$ is a continuous time Markov process on $\{0, 1, 2\}$ and find its generator.
- (b) Show that X is an irreducible Markov process.
- (c) Find the stationary distribution of X .
- (d) Find

$$\lim_{t \rightarrow \infty} \mathbb{P}[X(t) = 1 \mid X(0) = 0].$$

Q8 At time $n = 0$, an urn contains 1 blue ball and 1 red ball. At each time $n = 1, 2, 3, \dots$, a ball is chosen at random from the urn and returned to the urn together with a new ball of the same colour. Just after time n , there are $n + 2$ balls in the urn of which B_n are blue and R_n are red. Let

$$M_n = \frac{B_n}{B_n + R_n} = \frac{B_n}{n + 2}$$

be the proportion of blue balls in the urn after time n .

- (a) Show that M_n is a martingale sequence. You should specify the filtration.
- (b) Show that for an integer k with $1 \leq k \leq n + 1$,

$$\mathbb{P}[B_n = k] = \frac{1}{n + 1}.$$

- (c) Let T be the first time a blue ball is added to the urn. Compute

$$\mathbb{E}\left[\frac{1}{T + 2}\right]$$

carefully stating any theorems you use from the course.