EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH3281-WE01

Title:
Topology III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 (a) For a subset A of topological space X, define a *limit point* of A.

(b) Let $\mathbb{R}$ have the standard topology. Using your definition above, show that every real number is a limit point of the set $\mathbb{Q}$.

(c) As usual, let $\bar{A}$ denote the *closure* of A, and A° the *interior* of A. Define each of these terms.

(d) Among the following statements, only two are necessarily true for all subsets A, B of a topological space.

$$\overline{A \cap B} = \bar{A} \cap \bar{B}; \quad \overline{A \cup B} = \bar{A} \cup \bar{B}; \quad (A \cap B)^\circ = A^\circ \cap B^\circ; \quad (A \cup B)^\circ = A^\circ \cup B^\circ.$$

Give counterexamples for the ones that are not necessarily true.
(You are **not** asked for any proofs.)

Q2 (a) Let (X, τ) be a topological space, with sets $B \subset A \subset X$. We give A the subspace (induced) topology. Define carefully, in terms of τ, the meaning of the following statements:
i) (X, τ) is connected.
ii) B is open in A.
iii) B is not connected in A.

In Furstenberg's topology τ_F on $\mathbb{Z}$, a subset $U \subseteq \mathbb{Z}$ is defined to be open if for every $a \in U$ there exists a non-zero $d \in \mathbb{Z}$ with $a + d\mathbb{Z} \subseteq U$.

(b) Show that $(\mathbb{Z}, \tau_F)$ is not connected.

(c) Show that in $(\mathbb{Z}, \tau_F)$ the only nonempty connected sets are the singleton sets $\{x\}$.

Q3 (a) State what it means for two topological spaces to be homotopy equivalent.

(b) Consider the lists of upper- and lower-case letters below (in the given font!).

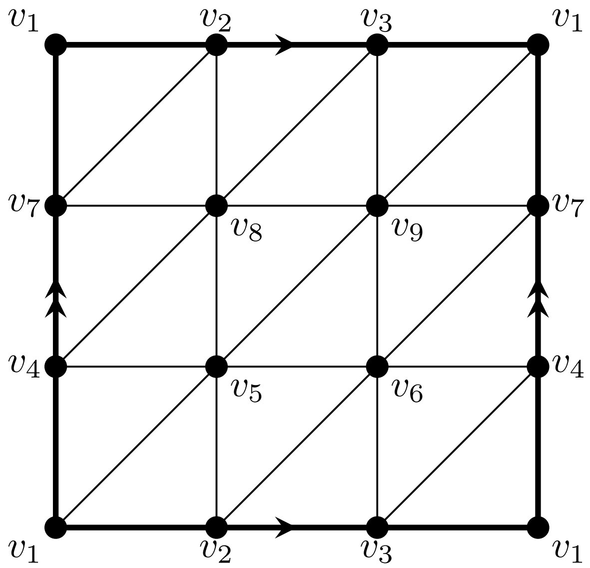
F G H I J
f g h i j

Viewing each letter as a subset of $\mathbb{R}^2$ equipped with the subspace topology, partition the upper-case list, the lower-case list and the combined list, respectively, into sets of homotopy-equivalent topological spaces. In particular, identify any letters from the upper-case list which are not homotopy equivalent to their lower-case counterparts. Briefly justify your answers, including by making reference to appropriate topological invariants wherever necessary.

(c) Let A be the annulus $A = \{z \in \mathbb{C} \mid 1 \le |z| < 2\}$, let S^1 be the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and let X be an arbitrary nonempty compact topological space. Prove that the product $X \times A$ is homotopy equivalent, but not homeomorphic, to the product $X \times S^1$.

Q4 (a) If K and L are finite simplicial complexes, state what it means for a map $f : K \to L$ to be a simplicial map.

(b) Let K be the 2-dimensional finite simplicial complex which triangulates the torus $T = S^1 \times S^1$ and is given (via the identifications indicated by the arrows on the sides of the square) by the diagram below, where the vertices are labelled $v_1, \ldots, v_9$.



Consider now the surjective simplicial map $f : K \to L$ determined by

$$f(v_i) = w_{i \bmod 3},$$

where L is a finite simplicial complex with vertices w_0, w_1, w_2.

(i) Sketch the simplicial complex L. State whether L triangulates a closed surface and, if so, identify that closed surface. Provide a brief justification for each part of your answer.

(ii) Compute the fundamental groups $\pi_1(K, v_1)$ and $\pi_1(L, w_1)$.

(iii) Deduce that the homomorphism $f_* : \pi_1(K, v_1) \to \pi_1(L, w_1)$ induced by f is surjective, but not an isomorphism.

SECTION B

Q5 (a) Give the definition of compactness for a topological space.

The Heine-Borel theorem states that, for a set $A \subseteq \mathbb{R}^n$,

$$A \text{ compact} \iff A \text{ closed and bounded.}$$

(b) Prove **one** direction, either $\Leftarrow$ or $\Rightarrow$, of Heine-Borel. You may use other results from the lectures, stating them clearly.

(c) Give an example of a metric space (M, d) and a set $A \subseteq M$, with A closed and bounded but not compact.

(d) For $n \geq 2$, determine which of the matrix groups $\mathrm{GL}_n(\mathbb{R}),\ \mathrm{SL}_n(\mathbb{R}),\ \mathrm{O}(n),\ \mathrm{SO}(n)$ are compact.

Q6 Let $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} |\ x_1^2 + \cdots + x_{n+1}^2 = 1\}$, G be the group $\{+1, -1\}$ with multiplication, and $G^{n+1} = \{(e_1, e_2, \ldots, e_{n+1}) \mid e_i \in \{-1, +1\}\}$ be the direct product of $n+1$ copies of G. We give each of S^n and G^{n+1} respectively the topology induced by the standard topology on $\mathbb{R}^{n+1}$. Then the map

$$\bullet : G^{n+1} \times S^n \to S^n \ \text{ given by } \ (e_1, \ldots, e_{n+1})(x_1, \ldots, x_{n+1}) \mapsto (e_1 x_1, \ldots, e_{n+1} x_{n+1})$$

defines an action of G^{n+1} on S^n.

(a) Consider the elements $(+1, +1, -1)$, $(+1, -1, -1)$ and $(-1, -1, -1)$ in the group G^3. Describe in simple geometric terms how each of these acts on S^2.

(b) Consider the orbits when G^{n+1} acts on S^n. For each $k \in \{1, \ldots, n+1\}$, specify an orbit with 2^k elements.

Let $f : S^n \to \mathbb{R}^{n+1}$ be given by $f((x_1, \ldots, x_{n+1})) = (x_1^2, \ldots, x_{n+1}^2)$.

(c) For $n = 2$, describe geometrically the image $f(S^2)$ of this map.

(d) As usual, we write S^n/G^{n+1} for the orbit space of this action, and define the maps

$$\pi : S^n \to S^n/G^{n+1} \ \text{ given by } \ \pi((x_1, \ldots, x_{n+1})) = [(x_1, \ldots, x_{n+1})]$$

$$\bar{f} : S^n/G^{n+1} \to \mathbb{R}^{n+1} \ \text{ given by } \ \bar{f}([(x_1, \ldots, x_{n+1})]) = f((x_1, \ldots, x_{n+1})),$$

so that $f = \bar{f} \circ \pi$. Show that $\bar{f}$ is both well-defined and injective.

(e) Show that $\bar{f}$ is a homeomorphism onto its image. You may use the following result from lectures: If X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

Q7 Let X be the connected, two-dimensional, finite simplicial complex given by $X = K \cup L$, where K and L are connected, two-dimensional, finite simplicial complexes and where the intersection $K \cap L$ is a single 0-simplex common to both K and L. Suppose, in addition, that K triangulates the Klein bottle and that L triangulates the topological space given by removing a small open disc from the real projective plane.

(a) Prove or disprove the statement that X is homeomorphic to a closed surface. Briefly justify any assertions you make.

(b) Compute the Euler characteristic of X, justifying any assertions you make.

(c) Compute the fundamental group $\pi_1(X)$.

[*You may assume knowledge of the fundamental group of the circle* S^1 *and of any contractible space, if necessary, but you should present as part of your answer a calculation of the fundamental group of any other space you use.*]

Q8 (a) Let S_1 and S_2 be two closed surfaces. State the definition of the connected sum $S_1 \# S_2$ of S_1 and S_2.

(b) Let $K\#\mathbb{P}$ be the connected sum of the Klein bottle K and the real projective plane $\mathbb{P}$.

(i) Compute the Euler characteristic $\chi(K\#\mathbb{P})$ of $K\#\mathbb{P}$, briefly justifying any formula you use.

(ii) Using $K\#\mathbb{P}$ as your starting point, explain how to obtain a closed, orientable surface Σ with Euler characteristic $\chi(\Sigma) = -2$ by attaching or removing discs, handles or crosscaps, as appropriate. You must give a brief explanation of the effect on the Euler characteristic of any operations you perform.