EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH3291-WE01

Title: Partial Differential Equations III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Consider the Cauchy problem associated to Burgers' equation

$$\begin{cases} \partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

Let $u_0(x) = \dfrac{1}{1+e^x}$ for $x \in \mathbb{R}$.

(a) Show that u_0 given above is bounded with bounded derivative.

(b) Show that there exists $t_c > 0$ such that the Cauchy problem (1) has a classical solution on $\mathbb{R} \times (0, t_c)$. Compute the precise value of t_c.

(c) What prevents this solution from being a global classical solution? Justify your answer.

Q2 We consider the following Cauchy problem for the unknown function $u : \mathbb{R}^n \to \mathbb{R}$,

$$\begin{cases} \boldsymbol{x} \cdot \nabla u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \mathbb{R}^n, \\ u(\boldsymbol{x}) = 1, & \boldsymbol{x} \in \Gamma, \end{cases} \tag{2}$$

where $n \in \mathbb{N}$, $n \geq 2$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and

$$\Gamma := \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : \; x_n = 0\}.$$

(a) Show that the map $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(\boldsymbol{x}) = \boldsymbol{x}$ is globally Lipschitz continuous. Compute its Lipschitz constant.

(b) Give a parametrisation for Γ and find all the points on it which are non-characteristic. Justify your answer.

(c) Find a classical solution to (2).

Q3 Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ be such that

$$\Delta u(\boldsymbol{x}) = \lambda u(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega$$

for some $\lambda \in \mathbb{R}$.

(a) Define the function
$$v(\boldsymbol{x}, t) := e^{\lambda t} u(\boldsymbol{x}).$$
Show that v satisfies the heat equation in $\Omega \times (0, +\infty)$, i.e.
$$v_t(\boldsymbol{x}, t) - \Delta v(t, \boldsymbol{x}) = 0, \quad x \in \Omega, \; t > 0.$$

(b) Assume in addition that $\lambda \geq 0$ and that u is non-negative and show that
$$\max_{\boldsymbol{x} \in \overline{\Omega}} u(\boldsymbol{x}) = \max_{\boldsymbol{x} \in \partial\Omega} u(\boldsymbol{x}).$$

Q4 Consider the following system

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{3}$$

where u_0 is a smooth function. Let u be a smooth solution for (3).

(a) Write the differential equation that the (spatial) Fourier transform of u, $\hat{u}(\xi,t)$, satisfies and consequently show that

$$\hat{u}(\xi,t) = \hat{u_0}\,(\xi)\, e^{i\xi^3 t}, \quad \xi \in \mathbb{R},\ t > 0.$$

You may assume that u_0, the solution u, and all their derivatives are smooth enough and go to zero fast enough at infinity so that you can use all the formulae from class for the Fourier transform.

(b) Assuming in addition that $u_0 \in L^2(\mathbb{R})$ and that $u(\cdot,t) \in L^2(\mathbb{R})$ for all $t > 0$ show that

$$\|u(\cdot,t)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$$

for all $t > 0$.

SECTION B

Q5 We consider the following Cauchy problem for the unknown function

$$\begin{cases} \partial_x u(x,y) + u(x,y)\partial_y u(x,y) = y, & (x,y) \in \mathbb{R}^2, \\ u(0,y) = y, & y \in \mathbb{R}. \end{cases} \tag{4}$$

We aim to use the method of characteristics to solve this.

(a) Determine the type of the PDE in (4) from the point of view of linearity. Justify your answer.

(b) Give a parametrisation of the Cauchy curve and find all non-characteristic points.

(c) Write down and solve the ODEs for the characteristics and for the solution along the characteristic. Using the solutions to these ODEs, write the solution to (4) and its domain of definition.

Q6 Consider the Cauchy problem associated to Burgers' equation

$$\begin{cases} \partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{5}$$

were $u_0 : \mathbb{R} \to \mathbb{R}$ is given by

$$u_0(x) = \begin{cases} 0, & x < -1, \\ x+1, & -1 \le x < 0, \\ -x+1, & 0 \le x < 1, \\ 0, & 1 \le x. \end{cases}$$

(a) Draw the graph of u_0 and in a separate figure sketch the characteristics.

(b) Find the first instance when the characteristics cross. At which location is this happening? Justify your answer.

(c) Discuss the need for shocks and rarefaction waves in order to construct a weak entropy solution to (5).

(d) Write down the ODEs satisfied by all the potential shocks. There is no need to solve these ODEs. [*Hint:* the corresponding left and right limits need to be carefully considered.]

Q7 Let $\Omega \subset \mathbb{R}^n$ be an open and connected set.

(a) Show that if $u, v \in C^2\left(\overline{\Omega}\right)$ then for any $\boldsymbol{x} \in \Omega$ and $r > 0$ such that $B_r(\boldsymbol{x}) \subset \Omega$

$$\left|\int_{B_r(\boldsymbol{x})} u(\boldsymbol{y})d\boldsymbol{y} - \int_{B_r(\boldsymbol{x})} v(\boldsymbol{y})d\boldsymbol{y}\right| \leq \sup_{\boldsymbol{z}\in\Omega} |u(\boldsymbol{z}) - v(\boldsymbol{z})| \, |B_r(\boldsymbol{x})| \,,$$

where $|B_r(\boldsymbol{x})|$ is the volume of the open ball of radius r centred at $\boldsymbol{x}$.

(b) Show that if $\{v_n\}_{n\in\mathbb{N}}$ is a sequence of $C^2\left(\overline{\Omega}\right)$ functions which are harmonic on Ω and if $v \in C^2\left(\overline{\Omega}\right)$ is a function such that

$$\lim_{n\to\infty}\left(\sup_{\boldsymbol{z}\in\Omega} |v_n(\boldsymbol{z}) - v(\boldsymbol{z})|\right) = 0,$$

then for any $\boldsymbol{x} \in \Omega$ and $r > 0$ such that $B_r(\boldsymbol{x}) \subset \Omega$ we must have that

$$v(\boldsymbol{x}) = \frac{1}{|B_r(\boldsymbol{x})|}\int_{B_r(x)} v(\boldsymbol{y})d\boldsymbol{y}.$$

You may use without proof the fact that uniform convergence of a sequence of functions implies pointwise convergence.

(c) Is $v(\boldsymbol{x})$ from sub-question (b) harmonic on Ω? Justify your answer.

Q8 For a given open and bounded domain with smooth boundary $\Omega \subset \mathbb{R}^n$ we define the functional

$$E : C^1\left(\overline{\Omega}\times[0,1]\right) \to \mathbb{R}$$

by

$$E[u] = \int_0^1 \int_\Omega \left(\frac{1}{2}u_t^2(\boldsymbol{x},t) + \frac{u^2(\boldsymbol{x},t)}{2}\left|\nabla u(\boldsymbol{x},t)\right|^2\right) d\boldsymbol{x}dt.$$

Let V be the set of all functions $\varphi \in C^1\left(\overline{\Omega}\times[0,1]\right)$ such that

$$\varphi(\boldsymbol{x},0) = \varphi(\boldsymbol{x},1) = 0, \quad \boldsymbol{x}\in\Omega,$$

$$\varphi(\boldsymbol{x},t) = 0, \quad \boldsymbol{x}\in\partial\Omega,\ t\in[0,1].$$

Assume that u is a smooth minimiser for $E[u]$. Show that for any $\varphi \in V$ we have that

$$\int_0^1\int_\Omega \left(-u_{tt}(\boldsymbol{x},t) - u(\boldsymbol{x},t)\left|\nabla u(\boldsymbol{x},t)\right|^2 - u^2(\boldsymbol{x})\,\Delta u(\boldsymbol{x},t)\right)\varphi(\boldsymbol{x},t)\,d\boldsymbol{x}dt = 0.$$