Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH3301-WE01

Title:
Mathematical Finance III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 Consider the market consisting of one risk-free asset with price dynamics $B_t = \left(\frac{5}{4}\right)^t$ and one risky asset whose price evolves with $S_0 = 24$, $u = \frac{3}{2}$ and $d = \frac{3}{4}$. Let $T = 2$.

(a) Prove that there is no arbitrage in this market, and find the risk-neutral measure.

(b) Calculate the fair prices at times $t = 0, 1, 2$, for a gap option with payoff

$$\Phi(S_T) = \begin{cases} S_T - 15 & \text{if } S_T > 20, \\ 0 & \text{if } S_T \leq 20. \end{cases}$$

(c) A broker offers to buy or sell the option in part (b) for £15. Is this a fair price? If it is not, describe in words how you would construct an arbitrage portfolio using this option (you do not have to actually construct the portfolio; "buy something and sell something else" is sufficient).

Q2 (a) State and prove the put-call parity theorem relating the fair prices at time 0 of a European call option with expiry date T and strike price K, and a European put option on the same stock with the same parameters. You should clearly define any other notation you use, and clearly state any results you use from the course.

(b) Hence or otherwise show that the difference in price between the two options is bounded above as

$$P - C \leq K.$$

Q3 (a) State the definition of a Brownian motion.

For the rest of the question, let $(W_t)_{t \geq 0}$ be a Brownian motion, and define

$$X_t := c\,(W_{2t+1} - W_1)$$

for $t \geq 0$.

(b) Prove, for a unique value of the constant $c > 0$ which you should determine, that $(X_t)_{t \geq 0}$ is also a Brownian motion.

(c) Choosing the constant c as determined in part (b), explain whether $(X_t)_{t \geq 0}$ is a martingale with respect to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by $(W_t)_{t \geq 0}$.

Q4 Let $(W_t)_{t \geq 0}$ be a Brownian motion, and define $X_t := \int_0^t s^4 \, dW_s$ for $t \geq 0$.

(a) For each $t \geq 0$, find $\mathbb{E}[X_t]$ and $\mathrm{Var}(X_t)$, and identify the distribution of X_t.

(b) Let $R = \int_0^1 X_s \, dW_s$. Find $\mathrm{Var}(R)$.

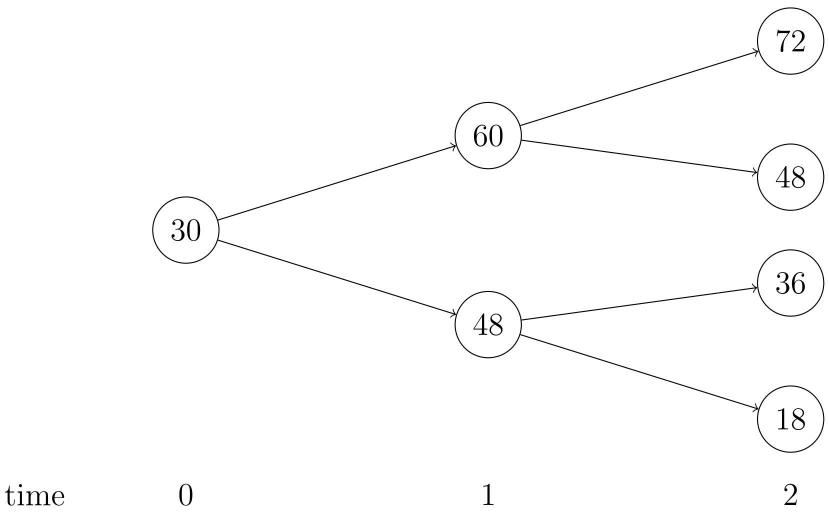
SECTION B

Q5 A *collar* is a trading strategy, in which the holder will:

- Buy one unit of the underlying risky asset,
- Buy one European put option, with expiry date T and strike price K,
- Short sell one European call option, with expiry date T and strike price $L > K$.

(a) Calculate and sketch the payoff of the collar option, as a function of S_T, the value of the risky asset at time T.

(b) Find the hedging portfolio for a collar option, with $T = 2$, $K = 24$ and $L = 60$, in the binomial market containing one risk-free asset with price dynamics $B_t = (1 + 0.2)^t$, and one risky asset whose price dynamics are shown in the tree below. Use your hedging portfolio to calculate the price of the option at every point.



Q6 Consider a discrete-time market containing two assets, as follows.

There is one risk-free asset, whose price dynamics are

$$B_t = \begin{cases} 1 & t = 0, \\ (1 + r_1) & t = 1, \\ (1 + r_1)(1 + r_2) & t = 2. \end{cases}$$

There is also one risky asset, with $S_0 = s$, which evolves according to a recombinant binomial model (with u and d fixed).

(a) Under which conditions on u, d, r_1 and r_2 does there exist a unique measure $\mathbb{Q}$ such that

$$\mathbb{E}_{\mathbb{Q}}[S_t] = B_t S_0$$

holds for $t = 0, 1, 2$?

Define $\mathbb{Q}$ (for example, by giving the values of q_u and q_d at each time-step).

(b) Using the risk-neutral valuation formula

$$V_0 = \frac{1}{B_T} \mathbb{E}_{\mathbb{Q}}[V_T],$$

find the fair price at time 0 for a European put option with expiry date $T = 2$ and strike price $K = 80$ in this market, if $s = 100$, $u = 1.2$, $d = 0.6$, $r_1 = 0.1$ and $r_2 = 0.05$.

Q7 Consider the stochastic differential equation

$$dX_t = (a - bX_t)\,dt + \sigma X_t\,dW_t, \qquad X_0 = 0,$$

where $a, b, \sigma > 0$ are constants, and $(W_t)_{t\geq 0}$ is a Brownian motion.

(a) Let $Y_t = \exp\left((b + \frac{\sigma^2}{2})t - \sigma W_t\right)$. Show that Y_t satisfies a stochastic differential equation of the form

$$\frac{dY_t}{Y_t} = c\,dt - \sigma\,dW_t,$$

and express the constant c in terms of b and σ.

(b) Let $Z_t := X_t Y_t$. By applying Itô's lemma, derive and simplify the stochastic differential equation satisfied by Z_t.

(c) By solving your stochastic differential equation from (b), or otherwise, show that for any fixed $t > 0$, X_t has the same distribution as

$$a \int_0^t \exp\left(-\left(b + \frac{\sigma^2}{2}\right) u + \sigma \widetilde{W}_u\right) du$$

where $(\widetilde{W}_u)_{u\in[0,t]}$ is some Brownian motion on $[0, t]$.

(d) Hence or otherwise, evaluate $\mathbb{E}[X_t]$.

Q8 Let $B_t = e^{rt}$ be the bond price at time t with risk-free interest rate $r = 0.05$, and let $(S_t)_{t\geq 0}$ be the stock price process following the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu_t\,dt + \sigma_t\,dW_t, \qquad S_0 = 2025,$$

where $(W_t)_{t\geq 0}$ is a Brownian motion (under the real world measure $\mathbb{P}$), and the functions μ and σ are given by

$$\mu_t = 0.3 + 0.1\sin(2\pi t), \qquad \sigma_t = 0.2\,[2 + \cos(2\pi t)] \qquad \text{for any } t \geq 0.$$

(a) Write down the stochastic differential equation satisfied by $Z_t := \log S_t$.

(b) Using the Itô isometry, or otherwise, find $\mathrm{Var}(Z_1)$.
(You may use the fact that $\int_0^1 \cos^2(2\pi t)\,dt = \frac{1}{2}$.)

(c) Let (a_t, b_t) be a portfolio and $V_t := a_t B_t + b_t S_t$ the value process. State the definitions for (a_t, b_t) to be a self-financing replicating portfolio of a contingent claim Φ at expiry time T.

(d) Let $\Pi_t(\Phi)$ be the no-arbitrage price of the contingent claim Φ at time t, and suppose that there exists a smooth function $F\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ such that $F(t, S_t) = \Pi_t(\Phi)$. By constructing a hedging portfolio, show that F satisfies a partial differential equation of the form

$$\begin{cases} \partial_t F + P(t,x)\partial_{xx}F + Q(t,x)\partial_x F - R(t,x)F = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad F(T,x) = \Phi(x), \end{cases}$$

and identify the functions $P(t, x)$, $Q(t, x)$ and $R(t, x)$.

END