Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH3391-WE01

Title:
Quantum Computing III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 A one-qubit system is described by the following density matrix,

$$\hat{\rho} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} .$$

(a) Verify that indeed the above is a valid density matrix.

(b) Show that this system is in a mixed state.

(c) Find a decomposition of $\hat{\rho}$ as a sum

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i \, , \tag{1}$$

where each ρ_i is a pure-state density matrix. Is this decomposition unique?

Q2 A self-adjoint operator $\hat{A}$ has normalised eigenstates $|1\rangle$, $|2\rangle$ and $|3\rangle$ with eigenvalues 1, 2, 3 respectively.

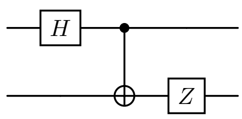
(a) For each of the two states below, determine the probability of measuring $A = 3$:

$$|\psi\rangle = \frac{1}{2}|1\rangle + \frac{1}{4}|2\rangle + \frac{1}{4}|3\rangle \, , \quad |\phi\rangle = \frac{3}{4}|1\rangle + \frac{1}{2}|2\rangle - \frac{1}{4}|3\rangle \, . \tag{2}$$

(b) Calculate the expectation value $\langle \hat{A} \rangle$ for the state $|\psi\rangle$ and for the state $|\phi\rangle$.

Q3 Given that {NOT, AND, OR, CNOT} is a universal gate set for classical computation, show that all classical computations can be performed reversibly by constructing reversible circuits realising each of these operations. Hence show that $P \subseteq BQP$.

Q4 Consider the circuit



Determine the action on the computational basis states. Determine the action on the states $|++\rangle$, $|+-\rangle$, $|-+\rangle$, $|--\rangle$.

SECTION B

Q5 Consider a 3-qubit system, in which Alice has two qubits and Bob has the remaining one. Assume that initially the state is given by

$$|\psi\rangle = \frac{1}{\sqrt{3}}\Big(|100\rangle + |010\rangle + |001\rangle \Big)\,. \tag{3}$$

(a) Compute the reduced density matrices $\hat{\rho}_A$ and $\hat{\rho}_B$.

(b) Compute the Von Neumann entropy S_A and S_B and check the consistency of your answer.

Q6 Consider a quantum device that claims to produce two-qubit systems in the Bell state

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\,. \tag{4}$$

(a) Compute the density matrix $\hat{\rho}$ for this pure state and write it in matrix form.

(b) Write down a generic expression for the expectation value of a measurement of an observable corresponding to an operator $\hat{W}$, in terms of $\hat{W}$ and a generic density matrix $\hat{\rho}$.

(c) Assume that $\hat{W}$ is given by the "witness operator",

$$\hat{W} = \frac{1}{2}\mathbb{I}_{4\times 4} - |\beta_{01}\rangle\langle\beta_{01}|\,. \tag{5}$$

Show that the expectation value is *negative* when the system is in the Bell state $|\beta_{01}\rangle$, but non-negative when the system is in the separable state $|01\rangle$.

(d) The above is a generic feature of witness operators: for all separable states, the expectation value is non-negative, and for at least one entangled state it is negative (you do *not* have to prove this).

Now suppose that due to noise, the actual output of the device is not $\hat{\rho}$ but a noisy mixed state $\hat{\rho}_{\text{noisy}}$,

$$\hat{\rho}_{\text{noisy}} = p\,\hat{\rho} + (1-p)\hat{N}\,, \tag{6}$$

where $\hat{\rho}$ is the density matrix computed in a) and p is a real parameter between 0 and 1 and $\hat{N}$ describes the form of the "noise". For $\hat{N} = \mathbb{I}/4$ ("white noise"), find the range of values of p for which the expectation value of the witness $\hat{W}$ will be smaller than zero, and the witness thus still detects the entanglement.

Q7 Given a unitary transformation acting on a two-qubit Hilbert space in the computational basis

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

(a) Write U as a product of unitaries U_i which each act non-trivially only on a two-dimensional subspace of the Hilbert space.

(b) Decompose each of the U_i in terms of controlled-unitary transformation and single-qubit unitaries.

(c) Construct a quantum circuit which realises U.

Q8 Consider a 3-qubit code given by

$$|\bar{0}\rangle := |001\rangle, \quad |\bar{1}\rangle := |110\rangle.$$

(a) Show that this code protects against single bit flip errors, describing how appropriate error syndromes are used to diagnose the errors and how the errors are corrected.

(b) Suppose that the system is instead subject to errors acting as Pauli Y on single qubits. Show that the same code is effective in protecting against these errors. [Recall $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Note that we are now considering a situation where we know the errors are only Y's. If both X and Y errors were possible, measuring the syndromes would tell us which qubit an error was on, but not which type.]

(c) Show that in the latter case, $\bar{Y} = Y_0 Y_1 Y_2$ provides a fault-tolerant implementation of Pauli Y on the logical qubit.