

EXAMINATION PAPER

Examination Session: May/June Year: 2025

Exam Code:

MATH3421-WE01

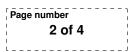
Title:

Bayesian Computation and Modelling III

Time:	2 hours		
Additional Material provided:	Formula sheet		
Materials Permitted:			
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.	

Instructions to Candidates:	Answer all questions.		
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.		
	Write your answer in the white-covered answer booklet with barcodes.		
	Begin your answer to each question on a new page.		

Revision:



SECTION A

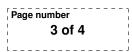
- Q1 Consider the simulation of values from a N(0, 1) distribution, truncated to the left at 0 and to the right at 1. The target density is $\pi(\theta) \propto \exp\{-\theta^2/2\}, 0 < \theta < 1$, and we wish to generate values of θ using a *Metropolis-Hastings independence sampler* based on Uniform U(0, 1) proposals.
 - (a) Write down and simplify the acceptance probability for a move from θ to ϕ .
 - (b) For what values of ϕ will the acceptance probability in (a) be 1? You should leave your answer in terms of θ .
 - (c) Given that the current value of the Markov chain simulated by this sampler is θ , find the probability that the chain moves (marginalised over the distribution of proposed values). You should leave your answer in terms of θ and the standard normal distribution function $\Phi(\cdot)$.
 - (d) The overall acceptance rate of this sampler (assuming that the chain is in equilibrium) is 0.92. Comment on this acceptance rate.
- Q2 The tobit model is widely used for random variables Y that are either exactly equal to zero or continuous and positive. Let $Y^*|\mu, \sigma^2 \sim N(\mu, \sigma^2)$ be a partially latent random variable; if $Y^* > 0$, we observe a value for $Y = Y^*$ but when $Y^* \leq 0$, we observe Y = 0. A random sample, y_1, \ldots, y_N , from the tobit model is therefore regarded as a set of N censored observations from the $N(\mu, \sigma^2)$ distribution, where zeros are treated as observations that are missing but known to be negative.
 - (a) Write down the joint likelihood for the model parameters μ and σ^2 with respect to N = n + m observations, assuming y_1, \ldots, y_n are strictly positive and the remaining m observations are equal to zero. You may use the standard normal distribution function $\Phi(\cdot)$ in your answer.
 - (b) Are both μ and σ^2 identifiable? Give your reasoning.
 - (c) An expert decides to model daily rainfall (in millimetres) in Durham as a random sample from the tobit model. They posit the following prior distribution for its parameters:

$$\mu \sim \mathcal{N}(a, 4),$$

$$\sigma^2 \sim \text{InvGamma}(2.5, 30),$$

where a is a constant to be chosen. Before seeing the data, the expert believes that Pr(Y = 0), i.e. the probability of a dry day, should be 0.6. Write down an integral equation that could be solved for a in order to capture the expert's judgement. Note that you need not solve the equation.

(d) Do you think this is a good model for daily rainfall? Give your reasoning.

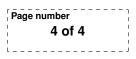


SECTION B

- **Q3** Consider a first order, homogeneous Markov chain $\underline{\theta}^{(n)}$, $n = 0, 1, \ldots$, where $\underline{\theta}^{(n)} = (\theta_1^{(n)}, \theta_2^{(n)})^T$, with continuous state-space \mathcal{S} . Let $\pi(\theta_1, \theta_2)$ be a density that is intended to be a stationary density of the Markov chain and let $p(\underline{\phi}|\underline{\theta})$ denote the transition density of the chain.
 - (a) Write down an integral equation satisfied by $\pi(\theta_1, \theta_2)$.
 - (b) Consider a deterministic scan Gibbs sampler with target density $\pi(\theta_1, \theta_2)$.
 - (i) Write down the form of the transition density $p(\underline{\phi}|\underline{\theta})$ of the resulting Markov chain, giving your answer in terms of full conditional densities of the form $\pi(\cdot|\cdot)$.
 - (ii) Using your answer to (a), show that the Markov chain defined by the deterministic scan Gibbs sampler has stationary density $\pi(\theta_1, \theta_2)$.
 - (c) Suppose that

$$\pi(\theta_1, \theta_2) \propto \theta_1^{-1} \exp\left\{2\theta_2 \log \theta_1 - 2\theta_2^2 - (\log \theta_1)^2\right\}, \quad \theta_1 > 0, \ \theta_2 \in \mathbb{R}.$$

- (i) Derive and identify all full conditional distributions for this bivariate target.
- (ii) The correlation between θ_1 and θ_2 is 0.54. Very briefly comment on the likely impact (if any) of this correlation on the mixing of the deterministic scan Gibbs sampler for this bivariate target.
- (iii) The marginal distribution of θ_2 is N(0, 1/2). Suggest a mechanism for direct sampling from $\pi(\theta_1, \theta_2)$.





Q4 The following model is proposed for the time to failure, Y, in units of thousands of hours, of light bulbs produced by a particular manufacturer:

$$Y|\alpha, \beta \sim \text{Gamma}(\alpha, \beta),$$

$$\alpha \sim \text{Gamma}(a_1, a_2),$$

$$\beta \sim \text{Gamma}(b_1, b_2).$$

(a) The Law of Total Expectation is given by $\mathbb{E}(\mathbf{Y}) = \mathbb{E} \{\mathbb{E}(\mathbf{Y}|\mathbf{X})\}$ for random vectors $\mathbf{X} = (X_1, \ldots, X_p)^T$ and $\mathbf{Y} = (Y_1, \ldots, Y_q)^T$. Use this to show that the mean of the preposterior distribution for Y is given by

$$\mathbb{E}(Y) = \frac{a_1 b_2}{a_2 (b_1 - 1)},$$

for $b_1 > 1$, and undefined otherwise.

- (b) Suppose that we decide to focus on the special case where α and β have exponential distributions (i.e. $a_1 = b_1 = 1$). Derive the preposterior density for Y, $\pi(y)$. Note that $\Gamma(a + 1) = a\Gamma(a)$.
- (c) We will soon have a random sample of 10 failure times.
 - (i) Draw a directed acyclic graph that represents the model for Y_1, \ldots, Y_{10} .
 - (ii) Hence or otherwise, explain whether you could calculate the evidence for this model using

$$\pi(\mathbf{y}) = \prod_{i=1}^{10} \pi(y_i),$$

where $\mathbf{y} = (y_1, \ldots, y_{10})^T$ denotes the random sample and $\pi(y_i)$ is an instance of the preposterior density for a single observation, derived in (b).

(d) Would you recommend the use of exponential priors for α and β in this model? Explain your reasoning.