Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH3471-WE01

Title:

Geometry of Mathematical Physics III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 (a) State the definition of $SU(2)$.

(b) Explain why $SU(2)$ is a group.

(c) Explain why $SU(2)$ is a Lie group.

Q2 Let V be the set of complex 2×2 matrices with the properties that $\mathrm{tr}M = 0$ and $M^\dagger = M$.

(a) Show that V is a vector space isomorphic to $\mathbb{R}^3$ and give a basis of V.

(b) For $g \in SU(2)$ and $M \in V$, show that $gMg^\dagger \in V$.

(c) Define an action of $g \in SU(2)$ on V by

$$F(g) : M \mapsto gMg^\dagger \, .$$

By showing that $F(g)$ preserves $\det M$, argue that $F(g)$ defines a group homomorphism from $SU(2)$ to $O(3)$.

Q3 Let $\mathbf{x}$ and $\mathbf{y}$ be Lorentz vectors with components x^μ and y^μ, let x_μ and y_μ be the components of the associated covectors, and let γ^μ be the Dirac matrices.

(a) How do x^μ, $x_\mu x^\mu$ and $x_\mu x^\nu$ behave under Lorentz transformations?

(b) For $\mathbf{x} = (1, 2, 0, 0)$ and $\mathbf{y} = (2, 1, 0, 0)$ compute $x^\mu x_\mu$, $y^\mu y_\mu$ and $x^\mu y_\mu$.

(c) Show that

$$(\gamma^\mu x_\mu)^2 = x_\mu x^\mu \; \mathbb{I}_{4\times 4} \, .$$

Q4 Consider a field theory with 2 complex scalar fields ϕ_1 ϕ_2 and action

$$S = \int d^4x \; \partial_\mu \bar{\phi}_1 \partial^\mu \phi_1 + \partial_\mu \bar{\phi}_2 \partial^\mu \phi_2 + \lambda \left(\phi_1 \phi_2 + \bar{\phi}_1 \bar{\phi}_2 \right)$$

for a real constant λ.

(a) Find the equations of motion for ϕ_1 and ϕ_2.

(b) Consider the $U(1)$ action

$$\phi_1 \to e^{iq_1\theta} \phi_1$$
$$\phi_2 \to e^{iq_2\theta} \phi_2$$

For which values of q_1 and q_2 is this a symmetry of S?

(c) Work out the conserved current associated to the $U(1)$ symmetry you found above and check that it is real.

SECTION B

Q5 Let V be the complex vector space of homogeneous polynomials in two variables z_1, z_2 of degree 2.

(a) Write a general element

$$P(z_1, z_2) = a_1 z_1^2 + a_2 z_1 z_2 + a_3 z_2^2 \,,$$

of V (for $a_1, a_2, a_3 \in \mathbb{C}$) as

$$P(z_1, z_2) = \sum_{j,k} z_j A_{jk} z_k$$

for a symmetric matrix A with components $A_{jk} = A_{kj}$ that you should find.

(b) Show that letting

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to g^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

for $g \in SU(2)$ defines a representation r of $SU(2)$ by working out the action of g on A.

(c) Describe the action of the associated Lie algebra representation ρ by working out the action of $\rho(i\sigma_j)$, for σ_j the Pauli matrices, on A.

(d) Verify that $\rho(i\sigma_j)$ is a Lie algebra homomorphism.

Q6 Let $SL(2,\mathbb{C})$ be the Lie group of complex 2×2 matrices of determinant 1.

(a) Find the Lie algebra elements γ_1 and γ_2 associated with the paths

$$p_1 : t \mapsto \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$
$$p_2 : t \mapsto \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix} .$$

(b) Show that

$$\exp\left(\sum_j \alpha_j \sigma_j\right)$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is in $SL(2,\mathbb{C})$ for arbitrary $\alpha_j \in \mathbb{C}$, and use this to describe a basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

(c) Work out $\exp(\alpha_1\sigma_1)$, $\exp(\alpha_2\sigma_2)$ and $\exp(\alpha_3\sigma_3)$ for $\alpha_j \in \mathbb{C}$.
hint:

$$\cosh \alpha = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!}$$
$$\sinh \alpha = \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!}$$

(d) Show that $G := \{\exp(\alpha_3\sigma_3), \alpha \in \mathbb{C}\}$ defines a subgroup of $SL(2,\mathbb{C})$. Find the invariant subspaces of the action

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \exp(\alpha_3\sigma_3) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

of G on $\mathbb{C}^2$.

Q7 Consider two inertial frames A and B which move relatively at a constant speed $\mathbf{v}_{AB} = (v_{AB}, 0, 0)$.

(a) Write down the boost taking us from frame A to frame B in terms of the rapidity λ_{AB} with $\tanh \lambda_{AB} = v_{AB}$.

(b) Assume there is a third frame C with relative motion to A given by a constant speed $\mathbf{v}_{AC} = (v_{AC}, 0, 0)$ and associated rapidity λ_{AC}. Find the Lorentz transformation Λ_{BC} which maps between the frames B and C.

(c) The above implies that frames B and C are also moving with constant velocity $\mathbf{v}_{BC}$ and rapidity λ_{BC}. Find λ_{BC} in terms of λ_{AB} and λ_{AC}.
Hint:

$$\cosh(x+y) = \cosh x \, \cosh y + \sinh x \, \sinh y \, ,$$
$$\sinh(x+y) = \cosh x \, \sinh y + \sinh x \, \cosh y \, .$$

(d) Show that for v_{AB} and v_{AC} both small compared to the speed of light c, we have approximately that

$$v_{BC} = v_{AC} - v_{AB} \, .$$

Q8 Consider a non-abelian gauge theory with gauge field $A_\mu = A_\mu^a t_a$ and a complex charged matter field $\phi = (\phi_1, \phi_2)$ in the defining representation of $SU(2)$ which transforms as

$$\phi \mapsto g(\mathbf{x})\phi$$

under gauge transformations $g(\mathbf{x}) \in SU(2)$.

(a) Show that $(\partial_\mu g(\mathbf{x}))\, g^{-1}(\mathbf{x}) + g(\mathbf{x})\,(\partial_\mu g^{-1}(\mathbf{x})) = 0$.

(b) Let the covariant derivative and gauge transformation of A_μ be given by

$$D_\mu \phi := \partial_\mu \phi - i g_{YM} A_\mu \phi$$

and

$$A_\mu \to g(\mathbf{x}) \left(A_\mu + i g_{YM}^{-1} \partial_\mu\right) g^{-1}(\mathbf{x})\,.$$

(Notice the unusual normalization leading to an appearance of the Yang-Mills coupling constant g_{YM}.)

Show that under a gauge transformation the covariant derivative transforms as

$$D_\mu \phi \mapsto g(\mathbf{x}) D_\mu \phi\,.$$

(c) Write the components $F_{\mu\nu}^a$ of the field strength

$$F_{\mu\nu} = F_{\mu\nu}^a t_a := \partial_\mu A_\nu - \partial_\nu A_\mu - i g_{YM} [A_\mu, A_\nu]$$

in terms of the components A_μ^a of the gauge field and the structure constants ${f_{ab}}^c$ of the Lie algebra of the gauge group given by

$$[t_a, t_b] = i {f_{ab}}^c t_c\ .$$

(d) Working in a normalization where $\mathrm{tr}(t_a t_b) = \frac{1}{2}\delta_{ab}$, write the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} \mathrm{tr}(F_{\mu\nu} F^{\mu\nu})$$

as a polynomial in g_{YM},

$$\mathcal{L} = \mathcal{L}_0 + g_{YM} \mathcal{L}_1 + g_{YM}^2 \mathcal{L}_2\ ,$$

and find explicit expressions for $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$ in terms of the components A_μ^a of the gauge field and the structure constants ${f_{ab}}^c$ of the Lie algebra.