Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH3491-WE01

Title:
Geometric Topology III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 (a) Explain what it means to say that two maps $f, g\colon X \to Y$ are homotopic, and prove that homotopy is an equivalence relation on maps.

(b) Let $D^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leqslant 1\}$ be the closed unit disc. Prove that, for any space X, any two maps $f, g\colon X \to D^2$ are homotopic. You may use any results from lectures about composites and homotopies of maps provided you state them carefully.

Q2 (a) Define the set of elements that make up the fundamental group $\pi_1(X, x_0)$ of a space X with base point x_0. Define the group product $*\colon \pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)$ on this set and show that it is well defined.

(b) If $f\colon (X, x_0) \to (Y, y_0)$ is a pointed map, define the homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Explain why it is well defined and prove that it satisfies $f_*(a * b) = f_*(a) * f_*(b)$ for elements $a, b \in \pi_1(X, x_0)$.

Q3 For the following statements, decide whether they are true or false. In either case, you have to justify your answer. If you think they are correct, you must give a sketch of proof. If they are incorrect, provide a counter-example. You are allowed to use properties of examples that appeared in the class or in the homework problems.

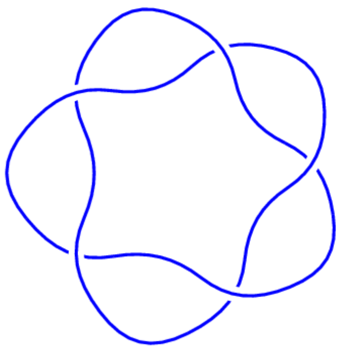
(a) The fundamental group of the complement of the unknot $U \subseteq S^3$ is a free abelian group isomorphic to $\mathbb{Z}$.

(b) The fundamental group of the complement of a link $L \subseteq S^3$ of 2 or more components is always non-abelian.

(c) A knot $K \subseteq S^3$ with trivial Alexander polynomial is the unknot.

(d) Suppose L_1 and L_2 are two links, each of two components, and suppose their complements have isomorphic fundamental groups. Then the links are isotopic.

Q4 Consider the following knot:



(a) Show that this is a torus knot by giving an appropriate sketch.

(b) Write down a presentation of the fundamental group of its complement. You do not need to prove that this is a correct presentation.

(c) Prove that the fundamental group of its complement is non-abelian by specifying a group homomorphism onto a non-abelian group.

SECTION B

Q5 (a) Say what it means for a map $p\colon Y \to X$ to be a covering map.

(b) Suppose G is a free group on r letters, and $H \subset G$ is an index n subgroup. Prove that H is also a free group, and deduce a relation between n, r and the rank of H. You may use without proof the Galois correspondence for covering spaces as covered in lectures.

(c) Use your result to find a rank 4 subgroup of the free group on 3 letters.

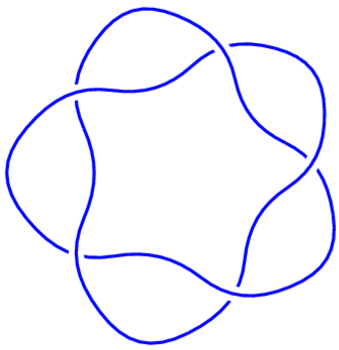
Q6 (a) Let X be the space given as the quotient of the unit square $[0,1] \times [0,1]$ by the following identifications.

$$\begin{aligned} (x,0) \sim (x,\tfrac{1}{2}) \sim (x,1) \quad & \text{for all } 0 \leqslant x \leqslant 1, \\ (0,y) \sim (1, y+\tfrac{1}{2}) \quad & \text{for all } 0 \leqslant y \leqslant \tfrac{1}{2} \end{aligned}$$

and let $Y \subset X$ be the subspace given by the (equivalence classes of) points $\{(x,y) \mid \frac{1}{3} \leqslant x \leqslant \frac{2}{3},\, 0 \leqslant y \leqslant 1\}$. Triangulate X so that Y is a subcomplex.

(b) Use your triangulation to compute $\pi_1(Y)$ and $\pi_1(X)$. Write $i\colon Y \to X$ for the inclusion map. Identify the homomorphism $i_*\colon \pi_1(Y) \to \pi_1(X)$. Is i_* an isomorphism?

Q7 Consider the same knot as in Question 4 above:

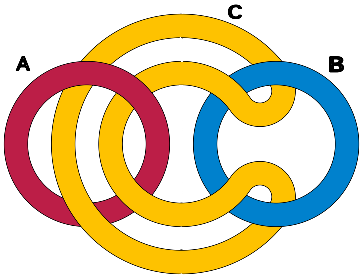


(a) Sketch a Seifert surface for this knot. If you find it more convenient, you can also use a different diagram of the same knot.

(b) Compute the Seifert matrix for this knot, and determine the Alexander polynomial of this knot.

(c) Determine the genus of this knot.

Q8 We consider the following 3-component link:



(a) Draw the Borromean rings and then show that the link pictured above is isotopic to the Borromean rings by drawing appropriate sketches. You will not need this result further on in this problem.

(b) Determine the fundamental group G of the complement of a 2-component unlink. It is enough to write down a presentation of this group.

(c) Prove that the group G contains a commutator $[x, y] = xyx^{-1}y^{-1}$ of two elements $x, y \in G$ which is non-trivial in G, that is $[x, y] \neq e$, where e denotes the identity element in G.

(d) The link consisting of the two components labeled A, B in the link above is a 2-component unlink. Fix some orientation on the component labeled C, and place a base-point somewhere on it. Prove that this represents a non-trivial element in G.

(e) Show that the link given by the components A, B, and C above is not equivalent to the 3-component unlink.