

# EXAMINATION PAPER

Examination Session: May/June

2025

Year:

Exam Code:

MATH4051-WE01

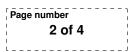
### Title:

# General Relativity IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within
	each section, all questions carry equal marks.
	Write your answer in the white-covered answer booklet with barcodes.
	Begin your answer to each question on a new page.

Revision:



#### SECTION A

**Q1** A two-dimensional spacetime has coordinates (t, x), and metric

$$ds^{2} = -dt^{2} + \frac{2}{x}dtdx + \frac{1}{x^{2}}dx^{2}.$$

New local coordinates are defined by  $u = t^2$ ,  $v = t + \ln x$ .

- (a) A vector field  $V^{\mu}$  has components  $V^{\mu} = (t, 0)$  with respect to the original coordinates. Calculate its components  $\tilde{V}^{\mu}$  with respect to the new coordinates.
- (b) Compute the metric in the new coordinates.
- **Q2** In a spacetime with coordinates (t, x, y) and metric  $ds^2 = -y^2 dt^2 + dx^2 + (1+x^2) dy^2$ , a curve is defined by  $x^{\mu} = (\lambda^{-1}, \lambda, \lambda^2)$  for  $\lambda \in (1, 2)$ .
  - (a) Compute the tangent vector  $V^{\mu}$  to the curve, and calculate its norm  $g_{\mu\nu}V^{\mu}V^{\nu}$ .
  - (b) Say whether the curve is timelike or spacelike, and calculate the proper time or proper length along the curve.
  - (c) Calculate the vector  $n^{\mu}$ , where  $n_{\mu}$  is normal to the surface t = x.
- Q3 Consider the following metric describing the spacetime near the surface of the Earth:

$$ds^{2} = -(1+2\Phi(r))dt^{2} + (1+2\Phi(r))^{-1}dr^{2} + r^{2}\left(d\theta^{2} + \sin^{2}\theta d\phi^{2}\right) + \frac{1}{2}\left(d\theta^{2} + \sin^{2}\theta d\phi^{2}\right) + \frac{1}{2}\left(d\theta^{2}$$

where  $\Phi(r) = -\frac{GM}{r}$ .

- (a) Obtain an approximation of the metric when  $GM \ll 1$ , to first non-trivial order in GM.
- (b) Consider an observer at rest on the surface of the Earth at radius  $r = R_1 > 2GM$ . How is the observer's proper time related to coordinate time t?
- (c) Consider another observer on a fixed position at  $r = R_2 > R_1$ . How is the observer's proper time related to coordinate time? If an event takes coordinate time  $\Delta t$ , which observer measures the largest proper time?
- Q4 Consider the following three dimensional black hole spacetime

$$ds^{2} = -(r^{2} - R^{2})dt^{2} + \frac{dr^{2}}{r^{2} - R^{2}} + r^{2}d\phi^{2}$$

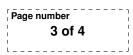
where R > 0 is a constant and  $\phi \in [0, 2\pi)$ . We consider the spacetime for r > R.

(a) Show that radial null paths on this spacetime can be written as

$$t = \pm f(r) + \text{const} \,,$$

where f(r) is a function you should determine. You can leave f(r) as an integral.

(b) Express the metric in terms of new coordinates  $(v, r, \phi)$ , where v is a coordinate such that ingoing (decreasing r) light rays obey v = constant.



#### SECTION B

Q5 Consider a spacetime with metric

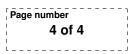
$$ds^{2} = -dt^{2} + z^{2}dx^{2} + \frac{dz^{2}}{z^{2}}.$$

- (a) Compute the Christoffel symbols for this spacetime.
- (b) Identify the Killing vectors for this spacetime, and find the associated conserved quantities for a geodesic  $x^{\mu}(\lambda)$ .
- (c) Show that geodesics with x not constant have  $z(\lambda) = \alpha \cosh \beta \lambda$ , where  $\alpha, \beta$  are constants which you should relate to the conserved quantities defined in the previous part. State any restrictions required on the conserved quantities. Find the form of  $t(\lambda)$ ,  $x(\lambda)$  for the geodesics.
- **Q6** A generalisation of a Killing vector is a Killing tensor, a symmetric tensor  $K_{\mu\nu}$  which satisfies

$$\nabla_{(\mu}K_{\nu\lambda)} = 0,$$

where the round brackets denote symmetrization.

- (a) Show that if  $K_{\mu\nu}$  is a Killing tensor, then  $P = K_{\mu\nu}V^{\mu}V^{\nu}$  is constant along an affinely-parametrized geodesic  $x^{\mu}(\lambda)$  with tangent vector  $V^{\mu}$ .
- (b) If  $X^{\mu}$  and  $Y^{\nu}$  are Killing vectors, show that  $K_{\mu\nu} = X_{(\nu}Y_{\nu)}$  is a Killing tensor. Does this lead to an independent conserved quantity?
- (c) Show that if  $K_{\mu\nu}$  is a Killing tensor, then  $\nabla_{\mu}K^{\mu}_{\ \lambda} = A\nabla_{\lambda}K^{\nu}_{\ \nu}$ , where A is a constant you should determine.
- (d) Determine the most general Killing tensor in two-dimensional flat space with metric  $ds^2 = dx^2 + dy^2$ .



**Q7** Consider the following action where the Ricci scalar R is coupled to a scalar field  $\phi(x)$  in the action as:

$$S[g_{\mu\nu},\phi] = \int d^4x \sqrt{-g} \left(\phi(x)^2 R + \frac{1}{2} \nabla_\mu \phi(x) \nabla^\mu \phi(x)\right) \,.$$

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Obtain the equations of motion for  $g_{\mu\nu}$ . You can use the following two variations:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,,$$

and

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_{\sigma} \left( g_{\mu\nu}\nabla^{\sigma}(\delta g^{\mu\nu}) - \nabla_{\lambda}(\delta g^{\sigma\lambda}) \right) \,.$$

When integrating by parts you can ignore boundary terms.

**Q8** A spacetime has metric

$$ds^{2} = -f(r)dt^{2} + \frac{dr^{2}}{f(r)} + r^{2} \left( d\theta^{2} + \sin^{2}\theta d\phi^{2} \right) ,$$

where

$$f(r) = \left(1 - \frac{2GM}{r}\right)^2$$

The geodesic equation for motion in the  $\theta = \pi/2$  plane reads

$$\dot{r}^2 = \left(\epsilon - \frac{L^2}{r^2}\right)f(r) + K^2,$$

where over-dots are derivatives with respect to an affine parameter, and L, K and  $\epsilon$  the conserved quantities along the geodesic:

$$L = r^2 \dot{\phi}, \quad K = f(r) \dot{t}, \quad \epsilon = g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}.$$

- (a) Does the metric have singularities? Justify your answer, but you do not need to distinguish if they are coordinate or curvature singularities.
- (b) An observer at a fixed position, with  $r = R_1$ , sends a light ray radially to another fixed observer at  $r = R_2 > R_1$ . How is the frequency measured by the observer at  $r = R_2$  related to the frequency emitted by the observer at  $r = R_1$ ?
- (c) Show that r is an affine parameter for a radial null geodesic. Tip: consider the geodesic equation in this case, and remember how affine parameters are related to one another.