Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH4051-WE01

Title: General Relativity IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 A two-dimensional spacetime has coordinates (t, x), and metric

$$ds^2 = -dt^2 + \frac{2}{x}dtdx + \frac{1}{x^2}dx^2.$$

New local coordinates are defined by $u = t^2$, $v = t + \ln x$.

(a) A vector field V^μ has components $V^\mu = (t, 0)$ with respect to the original coordinates. Calculate its components $\tilde{V}^\mu$ with respect to the new coordinates.

(b) Compute the metric in the new coordinates.

Q2 In a spacetime with coordinates (t, x, y) and metric $ds^2 = -y^2dt^2 + dx^2 + (1+x^2)\,dy^2$, a curve is defined by $x^\mu = (\lambda^{-1}, \lambda, \lambda^2)$ for $\lambda \in (1, 2)$.

(a) Compute the tangent vector V^μ to the curve, and calculate its norm $g_{\mu\nu}V^\mu V^\nu$.

(b) Say whether the curve is timelike or spacelike, and calculate the proper time or proper length along the curve.

(c) Calculate the vector n^μ, where n_μ is normal to the surface $t = x$.

Q3 Consider the following metric describing the spacetime near the surface of the Earth:

$$ds^2 = -(1 + 2\Phi(r))dt^2 + (1 + 2\Phi(r))^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,,$$

where $\Phi(r) = -\frac{GM}{r}$.

(a) Obtain an approximation of the metric when $GM \ll 1$, to first non-trivial order in GM.

(b) Consider an observer at rest on the surface of the Earth at radius $r = R_1 > 2GM$. How is the observer's proper time related to coordinate time t?

(c) Consider another observer on a fixed position at $r = R_2 > R_1$. How is the observer's proper time related to coordinate time? If an event takes coordinate time Δt, which observer measures the largest proper time?

Q4 Consider the following three dimensional black hole spacetime

$$ds^2 = -(r^2 - R^2)dt^2 + \frac{dr^2}{r^2 - R^2} + r^2d\phi^2\,,$$

where $R > 0$ is a constant and $\phi \in [0, 2\pi)$. We consider the spacetime for $r > R$.

(a) Show that radial null paths on this spacetime can be written as

$$t = \pm f(r) + \text{const}\,,$$

where $f(r)$ is a function you should determine. You can leave $f(r)$ as an integral.

(b) Express the metric in terms of new coordinates (v, r, ϕ), where v is a coordinate such that ingoing (decreasing r) light rays obey $v = $ constant.

SECTION B

Q5 Consider a spacetime with metric

$$ds^2 = -dt^2 + z^2 dx^2 + \frac{dz^2}{z^2}.$$

(a) Compute the Christoffel symbols for this spacetime.

(b) Identify the Killing vectors for this spacetime, and find the associated conserved quantities for a geodesic $x^\mu(\lambda)$.

(c) Show that geodesics with x not constant have $z(\lambda) = \alpha \cosh \beta\lambda$, where α, β are constants which you should relate to the conserved quantities defined in the previous part. State any restrictions required on the conserved quantities. Find the form of $t(\lambda)$, $x(\lambda)$ for the geodesics.

Q6 A generalisation of a Killing vector is a Killing tensor, a symmetric tensor $K_{\mu\nu}$ which satisfies

$$\nabla_{(\mu} K_{\nu\lambda)} = 0,$$

where the round brackets denote symmetrization.

(a) Show that if $K_{\mu\nu}$ is a Killing tensor, then $P = K_{\mu\nu} V^\mu V^\nu$ is constant along an affinely-parametrized geodesic $x^\mu(\lambda)$ with tangent vector V^μ.

(b) If X^μ and Y^ν are Killing vectors, show that $K_{\mu\nu} = X_{(\nu} Y_{\nu)}$ is a Killing tensor. Does this lead to an independent conserved quantity?

(c) Show that if $K_{\mu\nu}$ is a Killing tensor, then $\nabla_\mu K^\mu{}_\lambda = A \nabla_\lambda K^\nu{}_\nu$, where A is a constant you should determine.

(d) Determine the most general Killing tensor in two-dimensional flat space with metric $ds^2 = dx^2 + dy^2$.

Q7 Consider the following action where the Ricci scalar R is coupled to a scalar field $\phi(x)$ in the action as:

$$S[g_{\mu\nu}, \phi] = \int d^4x\sqrt{-g}\,\left(\phi(x)^2\,R + \frac{1}{2}\nabla_\mu\phi(x)\nabla^\mu\phi(x)\right)\,.$$

Obtain the equations of motion for $g_{\mu\nu}$. You can use the following two variations:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,,$$

and

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\sigma\left(g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})\right)\,.$$

When integrating by parts you can ignore boundary terms.

Q8 A spacetime has metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,,$$

where

$$f(r) = \left(1 - \frac{2GM}{r}\right)^2\,.$$

The geodesic equation for motion in the $\theta = \pi/2$ plane reads

$$\dot{r}^2 = \left(\epsilon - \frac{L^2}{r^2}\right)f(r) + K^2\,,$$

where over-dots are derivatives with respect to an affine parameter, and L, K and ϵ the conserved quantities along the geodesic:

$$L = r^2\dot{\phi}\,,\quad K = f(r)\dot{t}\,,\quad \epsilon = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\,.$$

(a) Does the metric have singularities? Justify your answer, but you do not need to distinguish if they are coordinate or curvature singularities.

(b) An observer at a fixed position, with $r = R_1$, sends a light ray radially to another fixed observer at $r = R_2 > R_1$. How is the frequency measured by the observer at $r = R_2$ related to the frequency emitted by the observer at $r = R_1$?

(c) Show that r is an affine parameter for a radial null geodesic. Tip: consider the geodesic equation in this case, and remember how affine parameters are related to one another.

END