EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH4061-WE01

Title:
Advanced Quantum Theory IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

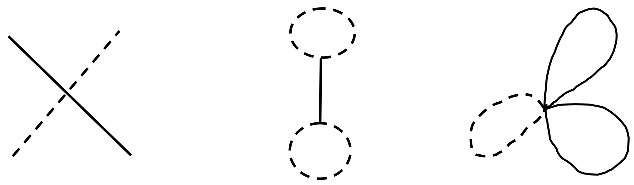
Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Consider the three Feynman diagrams below, for a theory with two real scalars φ_1 and φ_2. The solid lines represent the Feynman propagators of the field φ_1 and the dotted lines those of the field φ_2.



(1) **(2)** **(3)**

(a) For each diagram, write down the interaction part of the Lagrangian (up to a constant) from which the diagram follows.

(b) State at what order in perturbation theory each diagram enters, and the number of external legs for each diagram.

(c) Determine the symmetry factor of each diagram.

(d) State which diagrams (if any) are vacuum bubble diagrams, and which diagrams (if any) are disconnected.

Q2 Consider the following Lagrangian for $2n$ real scalar fields ϕ_1, ϕ_2:

$$\mathcal{L} = -\partial^\mu \phi_1 \partial_\mu \phi_2 - m^2 \phi_1 \phi_2 \ . \tag{1}$$

(a) Compute the Euler-Lagrange equation for the fields ϕ_1 and ϕ_2.

(b) Show that the Lagrangian is invariant under the infinitessimal transformation

$$\delta \phi_k = (-1)^k \epsilon \phi_k, \quad k = 1, 2 \ , \tag{2}$$

where ϵ is a small constant parameter.

(c) Compute the Noether current j^μ associated with the symmetry in part (b). Show that the current is conserved, using the results in part (a).

Q3 (a) The Hamiltonian for a Dirac fermion is

$$H = \int d^3x \, \bar{\Psi} \left(-i\gamma^i \partial_i + m \right) \Psi \tag{3}$$

where the index i runs over spatial directions. Show that in terms of creation and annihilation operators this is given by

$$H = \int \widetilde{dp} \sum_s \omega_p \left[b_s^\dagger (\boldsymbol{p}) \, b_s (\boldsymbol{p}) + d_s^\dagger (\boldsymbol{p}) \, d_s (\boldsymbol{p}) \right] + \text{constant} \tag{4}$$

where $\widetilde{dp} = \frac{d^3p}{\omega_p}$, $\omega_p = \sqrt{\boldsymbol{p}^2 + m^2}$, and you can neglect the divergent constant. You may also use the following formulae:

$$\Psi (x) = \sum_s \int \widetilde{dp} \left[b_s (\boldsymbol{p}) \, u_s (\boldsymbol{p}) \, e^{ip\cdot x} + d_s^\dagger (\boldsymbol{p}) \, v_s (\boldsymbol{p}) \, e^{-ip\cdot x} \right], \tag{5}$$

$$(\gamma^\mu p_\mu + m) \, u_s (\boldsymbol{p}) = (-\gamma^\mu p_\mu + m) \, v_s (\boldsymbol{p}) = 0, \tag{6}$$

$$u_s^\dagger(\boldsymbol{p}) v_{s'}(-\boldsymbol{p}) = v_{s'}^\dagger(\boldsymbol{p}) u_s(-\boldsymbol{p}) = 0, \tag{7}$$

$$u_{s'}^\dagger(\boldsymbol{p}) u_s(\boldsymbol{p}) = v_{s'}^\dagger(\boldsymbol{p}) v_s(\boldsymbol{p}) = 2\omega_p \delta_{ss'}, \tag{8}$$

$$\left\{ b_s(\boldsymbol{p}), b_{s'}^\dagger(\boldsymbol{p}') \right\} = \left\{ d_s(\boldsymbol{p}), d_{s'}^\dagger(\boldsymbol{p}') \right\} = (2\pi)^3 \, \delta^3 \left(\boldsymbol{p} - \boldsymbol{p}' \right) 2\omega_p \delta_{ss'}, \tag{9}$$

where $p^\mu = (\omega_p, \boldsymbol{p})$.

(b) The momentum operator of a Dirac fermion is given by

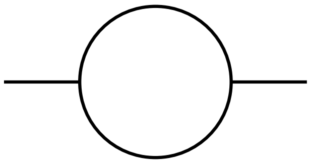
$$P_i = \int d^3x \, \Psi^\dagger \left(-i\partial_i \right) \Psi. \tag{10}$$

Derive a formula for this in terms of creation and annihilation operators. You may neglect a divergent constant, as before.

Q4 Consider the ϕ^3 theory in $d = 4$ spacetime dimensions:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{3!}\lambda\phi^3. \tag{11}$$

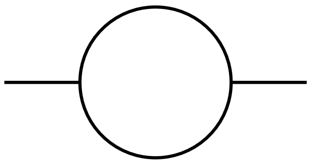
(a) Show that the 1-loop correction to the self-energy which arises from the following amputated diagram is given by



$$= \frac{1}{2}\lambda^2 \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 + m^2)\left((l+p)^2 + m^2\right)}, \tag{12}$$

where the external momentum p^μ is off-shell.

(b) Evaluate the loop integral using a cut-off Λ and show that for $p^2 = 0$ and $\Lambda^2 \gg m^2$, it is given by



$$= \frac{i\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - 1\right]. \tag{13}$$

It may be useful to recall that the area of a unit 3-sphere is $2\pi^2$. The following formulae may also be useful:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{\left(xA + (1-x)B\right)^2}, \tag{14}$$

$$\int_0^z \frac{x^3 dx}{\left(x^2 + y^2\right)^2} = \frac{1}{2}\left(\ln\left(\frac{z^2 + y^2}{y^2}\right) - \frac{z^2}{z^2 + y^2}\right). \tag{15}$$

SECTION B

Q5 The action for a single real free scalar field ϕ takes the form

$$S = \int d^4x \left(-\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \right) \tag{16}$$

The Fourier expansion for the field operator takes the form (recall we drop the hats from operators to lighten the notation):

$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}} e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} \right) \tag{17}$$

(a) Give the expression for $\omega_{\vec{k}}$ in terms of the momentum $\vec{k}$ and the mass m.

(b) Compute the Fourier expansion of the conjugate momentum operator π in this theory.

(c) Write $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ in terms of ϕ and π.

(d) Given $[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y})$, compute the equal-time commutator $[a_{\vec{k}}, a^\dagger_{\vec{k}'}]$.

Q6 Consider a theory of two real scalar fields ϕ and φ with action

$$S = \int d^4x \left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2}m_2^2\varphi^2 - \frac{\lambda}{2}\phi^2\varphi \right) \, , \tag{18}$$

where λ is the coupling constant.

(a) Write out the Feynman rules for time ordered vacuum expectation values in this theory. Note that there are two scalar fields, each with their own Feynman propagator. Use a solid line for the propagator of ϕ and a dotted line for the propagator of φ.

(b) Consider the vacuum-to-vacuum amplitude

$$_0\langle 0|T\left\{ \exp\left[-i\frac{\lambda}{2}\int d^4x' \, \phi_I^2\left(x'\right)\varphi_I\left(x'\right)\right]\right\}|0\rangle_0 \, , \tag{19}$$

where $|0\rangle_0$ is the free theory vacuum. Expand this up to and including order λ^2, and apply Wick's theorem to find expressions in terms of integrals over the Feynman propagators .

(c) Draw the Feynman diagrams for the processes described in (b). Apply the Feynman rules from part (a) to compute the amplitude and verify that you get the same results as you did in (b).

(d) Draw the Feynman diagrams for the processes

$$\langle 0|T\left\{\phi\left(x\right)\phi\left(y\right)\right\}|0\rangle = \frac{_0\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\frac{\lambda}{2}\int d^4x' \, \phi_I^2(x')\varphi_I(x')\right]\right\}|0\rangle_0}{_0\langle 0|T\left\{\exp\left[-i\frac{\lambda}{2}\int d^4x' \, \phi_I^2(x')\varphi_I(x')\right]\right\}|0\rangle_0}, \tag{20}$$

$$\langle 0|T\left\{\phi\left(x\right)\varphi\left(y\right)\right\}|0\rangle = \frac{_0\langle 0|T\left\{\phi_I(x)\varphi_I(y)\exp\left[-i\frac{\lambda}{2}\int d^4x' \, \phi_I^2(x')\varphi_I(x')\right]\right\}|0\rangle_0}{_0\langle 0|T\left\{\exp\left[-i\frac{\lambda}{2}\int d^4x' \, \phi_I^2(x')\varphi_I(x')\right]\right\}|0\rangle_0}, \tag{21}$$

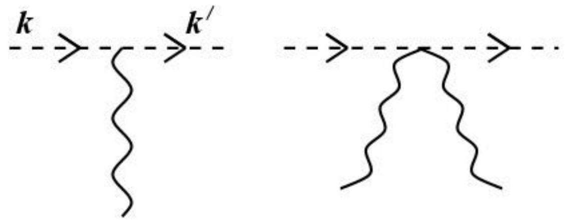
$$\langle 0|T\left\{\varphi\left(x\right)\varphi\left(y\right)\right\}|0\rangle = \frac{_0\langle 0|T\left\{\varphi_I(x)\varphi_I(y)\exp\left[-i\frac{\lambda}{2}\int d^4x' \, \phi_I^2(x')\varphi_I(x')\right]\right\}|0\rangle_0}{_0\langle 0|T\left\{\exp\left[-i\frac{\lambda}{2}\int d^4x' \, \phi_I^2(x')\varphi_I(x')\right]\right\}|0\rangle_0}, \tag{22}$$

up to and including order λ^2.

Q7 The Lagrangian for scalar QED is given by

$$\mathcal{L} = -\left(D^{\mu}\phi\right)^{\dagger} D_{\mu}\phi - m^2\phi^{\dagger}\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{23}$$

where ϕ is a complex scalar field and $D_{\mu}\phi = \partial_{\mu}\phi - ieA_{\mu}\phi$. The interaction vertices are depicted below and are given by $ie\left(k + k'\right)_{\mu}$ and $-2ie^2\eta_{\mu\nu}$, respectively. The



arrows indicate the flow of charge.

(a) Show that the action is invariant under gauge transformations

$$\phi \to e^{-ie\Omega(x)}, \;\; A_{\mu} \to A_{\mu} - \partial_{\mu}\Omega(x). \tag{24}$$

(b) Show that the tree-level Compton amplitude $\tilde{e}_1^{-}\gamma_2 \to \gamma_3\tilde{e}_4^{-}$ is given by

$$i\mathcal{M} = 4ie^2\left[\frac{k_1\cdot\epsilon_2 k_4\cdot\epsilon_3}{\left(k_1+k_2\right)^2+m^2} + \frac{k_1\cdot\epsilon_3 k_4\cdot\epsilon_2}{\left(k_1-k_3\right)^2+m^2} - \frac{1}{2}\epsilon_2\cdot\epsilon_3\right], \tag{25}$$

where $\tilde{e}^{-}$ is the state annihilated by the field ϕ and created by $\phi^{\dagger}$, and the subscripts label the external legs.

(c) Show that the amplitude vanishes if we replace the polarisation of either photon with its momentum.

Q8 (a) Recall that the Lagrangian for ϕ^4 theory is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g\phi^4. \tag{26}$$

Draw all connected diagrams with no external sources and at most two interaction vertices and indicate their symmetry factors.

(b) Consider the following integral:

$$e^{\tilde{w}(g,J)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}x^2 + \frac{1}{4!}gx^4 + Jx\right). \tag{27}$$

This integral does not converge, but can be used to generate a joint power series in g and J:

$$\tilde{w}(g,J) = \sum_{V=0}^{\infty}\sum_{E=0}^{\infty} C_{V,E}\, g^V J^E. \tag{28}$$

Show that

$$e^{\tilde{w}(g,J)} = \sum_{V=0}^{\infty}\frac{1}{V!}\left[\frac{g}{4!}\left(\frac{\partial}{\partial J}\right)^4\right]^V \sum_{P=0}^{\infty}\frac{1}{P!}\left[\frac{1}{2}J^2\right]^P. \tag{29}$$

(c) Recall that in ϕ^4 theory there is an analogous quantity

$$\begin{aligned}\exp\tilde{W}(g,J) = &\sum_{V=0}^{\infty}\frac{1}{V!}\left[\frac{g}{4!}\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4\right]^V \\ &\times\sum_{P=0}^{\infty}\frac{1}{P!}\left[\frac{i}{2}\int d^4y d^4z J(y)\Delta(y-z)J(z)\right]^P,\end{aligned} \tag{30}$$

where $\tilde{W}(g,J)$ is the sum over all connected diagrams including symmetry factors. Use this to show that

$$C_{V,E} = \sum_I \frac{1}{S_I}, \tag{31}$$

where the sum is over all connected diagrams with E sources and V vertices in ϕ^4 theory and S_I is the symmetry factor for each diagram.

(d) Determine the coefficient $C_{1,0}$ by expanding the integral in (27) to the appropriate order in g and J, and compare it with your solution in part (a). The following integrals may be useful:

$$\frac{1}{\sqrt{2\pi}}\int dx e^{-x^2} = 1, \quad \frac{1}{\sqrt{2\pi}}\int dx e^{-x^2}x^4 = 3. \tag{32}$$