EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
|---|---|---|
| May/June | 2025 | MATH40920-WE01 |

| Title: |
|---|
| Mathematical Finance V |

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| Time: | 3 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | Yes | Models Permitted: Casio FX83 series or FX85 series. |

| | |
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| Instructions to Candidates: | Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks.<br>Write your answer in the white-covered answer booklet with barcodes.<br>Begin your answer to each question on a new page. |

| Revision: | |
|---|---|

ED01/2025

## SECTION A

**Q1** Consider the market consisting of one risk-free asset with price dynamics $B_t = \left(\frac{5}{4}\right)^t$ and one risky asset whose price evolves with $S_0 = 24$, $u = \frac{3}{2}$ and $d = \frac{3}{4}$. Let $T = 2$.

(a) Prove that there is no arbitrage in this market, and find the risk-neutral measure.

(b) Calculate the fair prices at times $t = 0, 1, 2$, for a gap option with payoff

$$\Phi(S_T) = \begin{cases} S_T - 15 & \text{if } S_T > 20, \\ 0 & \text{if } S_T \leq 20. \end{cases}$$

(c) A broker offers to buy or sell the option in part (b) for £15. Is this a fair price? If it is not, describe in words how you would construct an arbitrage portfolio using this option (you do not have to actually construct the portfolio; "buy something and sell something else" is sufficient).

**Q2** (a) State the definition of a Brownian motion.

For the rest of the question, let $(W_t)_{t \geq 0}$ be a Brownian motion, and define

$$X_t := c\,(W_{2t+1} - W_1)$$

for $t \geq 0$.

(b) Prove, for a unique value of the constant $c > 0$ which you should determine, that $(X_t)_{t \geq 0}$ is also a Brownian motion.

(c) Choosing the constant $c$ as determined in part (b), explain whether $(X_t)_{t \geq 0}$ is a martingale with respect to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by $(W_t)_{t \geq 0}$.

**Q3** Let $(W_t)_{t \geq 0}$ be a Brownian motion, and define $X_t := \int_0^t s^4\, dW_s$ for $t \geq 0$.

(a) For each $t \geq 0$, find $\mathbb{E}[X_t]$ and $\text{Var}(X_t)$, and identify the distribution of $X_t$.

(b) Let $R = \int_0^1 X_s\, dW_s$. Find $\text{Var}(R)$.

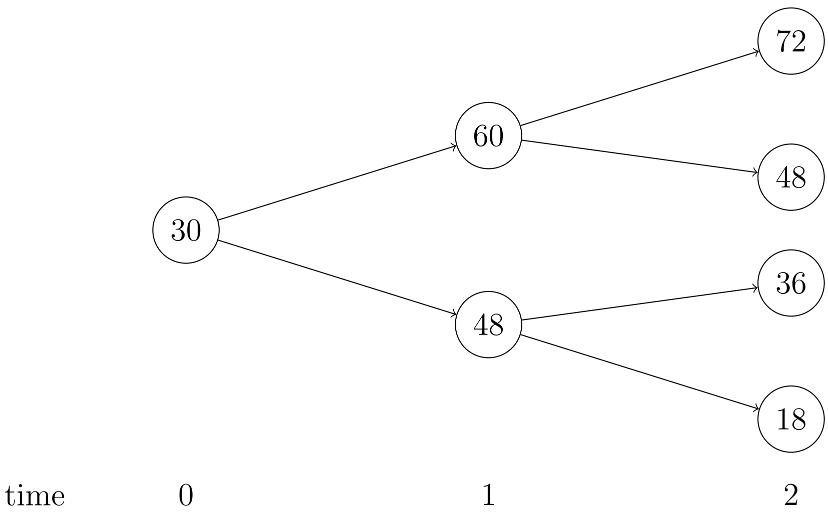
## SECTION B

**Q4** A *collar* is a trading strategy, in which the holder will:

- Buy one unit of the underlying risky asset,
- Buy one European put option, with expiry date $T$ and strike price $K$,
- Short sell one European call option, with expiry date $T$ and strike price $L > K$.

(a) Calculate and sketch the payoff of the collar option, as a function of $S_T$, the value of the risky asset at time $T$.

(b) Find the hedging portfolio for a collar option, with $T = 2$, $K = 24$ and $L = 60$, in the binomial market containing one risk-free asset with price dynamics $B_t = (1 + 0.2)^t$, and one risky asset whose price dynamics are shown in the tree below. Use your hedging portfolio to calculate the price of the option at every point.



**Q5** Consider a discrete-time market containing two assets, as follows.

There is one risk-free asset, whose price dynamics are

$$B_t = \begin{cases} 1 & t = 0, \\ (1 + r_1) & t = 1, \\ (1 + r_1)(1 + r_2) & t = 2. \end{cases}$$

There is also one risky asset, with $S_0 = s$, which evolves according to a recombinant binomial model (with $u$ and $d$ fixed).

(a) Under which conditions on $u, d, r_1$ and $r_2$ does there exist a unique measure $\mathbb{Q}$ such that

$$\mathbb{E}_{\mathbb{Q}}[S_t] = B_t S_0$$

holds for $t = 0, 1, 2$?

Define $\mathbb{Q}$ (for example, by giving the values of $q_u$ and $q_d$ at each time-step).

(b) Using the risk-neutral valuation formula

$$V_0 = \frac{1}{B_T} \mathbb{E}_{\mathbb{Q}}[V_T],$$

find the fair price at time 0 for a European put option with expiry date $T = 2$ and strike price $K = 80$ in this market, if $s = 100$, $u = 1.2$, $d = 0.6$, $r_1 = 0.1$ and $r_2 = 0.05$.

**Q6** Consider the stochastic differential equation

$$dX_t = (a - bX_t)\,dt + \sigma X_t\,dW_t, \qquad X_0 = 0,$$

where $a, b, \sigma > 0$ are constants, and $(W_t)_{t\geq 0}$ is a Brownian motion.

(a) Let $Y_t = \exp\big((b + \frac{\sigma^2}{2})t - \sigma W_t\big)$. Show that $Y_t$ satisfies a stochastic differential equation of the form

$$\frac{dY_t}{Y_t} = c\,dt - \sigma\,dW_t,$$

and express the constant $c$ in terms of $b$ and $\sigma$.

(b) Let $Z_t := X_t Y_t$. By applying Itô's lemma, derive and simplify the stochastic differential equation satisfied by $Z_t$.

(c) By solving your stochastic differential equation from (b), or otherwise, show that for any fixed $t > 0$, $X_t$ has the same distribution as

$$a\int_0^t \exp\left(-\left(b + \frac{\sigma^2}{2}\right)u + \sigma\widetilde{W}_u\right)du$$

where $(\widetilde{W}_u)_{u\in[0,t]}$ is some Brownian motion on $[0,t]$.

(d) Hence or otherwise, evaluate $\mathbb{E}[X_t]$.

**Q7** Let $B_t = e^{rt}$ be the bond price at time $t$ with risk-free interest rate $r = 0.05$, and let $(S_t)_{t\geq 0}$ be the stock price process following the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu_t\,dt + \sigma_t\,dW_t, \qquad S_0 = 2025,$$

where $(W_t)_{t\geq 0}$ is a Brownian motion (under the real world measure $\mathbb{P}$), and the functions $\mu$ and $\sigma$ are given by

$$\mu_t = 0.3 + 0.1\sin(2\pi t), \qquad \sigma_t = 0.2\,[2 + \cos(2\pi t)] \qquad \text{for any } t \geq 0.$$

(a) Write down the stochastic differential equation satisfied by $Z_t := \log S_t$.

(b) Using the Itô isometry, or otherwise, find $\mathrm{Var}(Z_1)$.
(You may use the fact that $\int_0^1 \cos^2(2\pi t)\,dt = \frac{1}{2}$.)

(c) Let $(a_t, b_t)$ be a portfolio and $V_t := a_t B_t + b_t S_t$ the value process. State the definitions for $(a_t, b_t)$ to be a self-financing replicating portfolio of a contingent claim $\Phi$ at expiry time $T$.

(d) Let $\Pi_t(\Phi)$ be the no-arbitrage price of the contingent claim $\Phi$ at time $t$, and suppose that there exists a smooth function $F\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ such that $F(t, S_t) = \Pi_t(\Phi)$. By constructing a hedging portfolio, show that $F$ satisfies a partial differential equation of the form

$$\begin{cases} \partial_t F + P(t,x)\partial_{xx}F + Q(t,x)\partial_x F - R(t,x)F = 0, \\ \hfill F(T,x) = \Phi(x), \end{cases}$$

and identify the functions $P(t,x)$, $Q(t,x)$ and $R(t,x)$.

## SECTION C

**Q8** (a) Write down the usual unbiased estimators for the mean and variance of a random variable $X$ constructed from the sample $X_1, X_2, \ldots, X_M$. Under what conditions on the random variables $X_1, X_2, \ldots, X_M$ are each of these estimators unbiased?

(b) Give an approximate 95% confidence interval $[L, U]$ for $\mathbb{E}[X]$, assuming that $M$ is large. Explain why

$$\mathbb{P}\big(\mathbb{E}[X] \in [L, U]\big) \approx 0.95$$

if $M$ is large.

You may find some of the following values of the standard normal distribution function helpful.

| $x$ | 1.645 | 1.960 | 2.330 |
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| $N(x)$ | 0.95 | 0.975 | 0.99 |

(c) Suppose that $S_t$ is the price of a risky asset which evolves randomly according to a geometric Brownian motion with drift $\mu$ and volatility $\sigma$, and that interest is compounded continuously at rate $r$.

Suppose that you also have a sequence of random variables $\xi_1, \xi_2, \ldots, \xi_M$ which are independently sampled from a standard Normal distribution.

Describe how these Normal samples can be used in the previous part to find a confidence interval for the payoff of a contingent claim, given by $X = \Lambda(S_T)$. (You can assume that $\Lambda$ is a continuous function.) You might do this by setting $X_i = f(\xi_i)$, where $f$ is some function to be defined.