Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH41220-WE01

Title: Analysis V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Let $E \subset \mathbb{R}$. Let $\mu^*(E)$ denote the Lebesgue outer measure of E.

(a) State what it means for a set $E \subset \mathbb{R}$ to be Lebesgue measurable.

(b) Suppose that $\mu^*(E) = 0$. Prove that E is Lebesgue measurable and hence prove that any subset E' of E is Lebesgue measurable with Lebesgue measure zero. Remember to state the names of any properties of μ^* that you use.

Q2 Let $E \subset \mathbb{R}$ be Lebesgue measurable.

(a) State the definition of $L^p(E)$ for $1 \leq p < \infty$.

(b) Let $f \in L^4(E)$. Prove that there exist $g \in L^2(E)$ and $h \in L^6(E)$ such that $f = g + h$.

Q3 Let X be an inner-product space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$.

(a) State and prove the Parallelogram Identity.

(b) Recall that the space ℓ_∞ is the linear space of all bounded sequences of real numbers, i.e., $x = (x_n)_{n \in \mathbb{N}} \in \ell_\infty$ if $x_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Also recall that the norm on ℓ_∞ is $\|x\|_{\ell_\infty} = \sup_{n \in \mathbb{N}} |x_n|$. Prove that there does not exist an inner product $\langle \cdot, \cdot \rangle_\infty$ on ℓ_∞ such that $\| \cdot \|_{\ell_\infty} = \sqrt{\langle \cdot, \cdot \rangle_\infty}$.

SECTION B

Q4 (a) Let $g_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be defined by

$$g_n(x) = n^2 \mathbb{1}_{[\frac{1}{n}, \frac{2}{n}]}(x),$$

and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 0$. The sequence of functions $(g_n)_{n=1}^{\infty}$ converges pointwise to g as $n \to \infty$. Does the Monotone Convergence Theorem apply in this case? Justify your response and compute the integrals $\int g_n$, $\int g$.

(b) Let $f_n \in \mathcal{M}^+$, $n \in \mathbb{N}$.

(i) State Fatou's Lemma.

(ii) Let $f \in \mathcal{M}^+$. Suppose that $(f_n)_{n=1}^{\infty}$ converges pointwise to f on $\mathbb{R}$ as $n \to \infty$ and that $\lim_{n\to\infty} \int f_n = \int f$ where $\int f < \infty$. Let $E \subset \mathbb{R}$ be Lebesgue measurable. Prove that

$$\int_{\mathbb{R}\setminus E} f \leq \int f - \limsup_{n\to\infty} \int_E f_n,$$

and hence that

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Q5 Let $E \subset \mathbb{R}$ be Lebesgue measurable and $f : E \to \mathbb{R} \cup \{\infty\}$.

(a) State what it means for f to be Lebesgue measurable.

(b) Prove that if f is Lebesgue measurable, then for each open set $U \subset \mathbb{R}$ the set $f^{-1}(U) = \{x \in E : f(x) \in U\}$ is Lebesgue measurable. [You may use the *Structure of Open Sets* theorem from lectures].

(c) Let $h : E \to \mathbb{R}$ be Lebesgue measurable and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that $g \circ h : E \to \mathbb{R}$, defined as $(g \circ h)(x) = g(h(x))$, is Lebesgue measurable. [You may use the result that if g is continuous then for any open set $U \subset \mathbb{R}$, $g^{-1}(U) = \{x \in \mathbb{R} : g(x) \in U\}$ is open].

Q6 Let $E \subset \mathbb{R}$ be Lebesgue measurable. Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $g \in L^q(E)$ and define $T_g : L^p(E) \to \mathbb{R}$ by

$$T_g(f) = \int_E fg \quad \text{for all } f \in L^p(E).$$

(a) Prove that T_g is a bounded linear functional. Remember to state the names of any inequalities from lectures that you use.

(b) Recall that

$$\|T_g\|_* = \sup_{f \in L^p(E), \|f\|_{L^p} \leq 1} |T_g(f)|.$$

Prove that $\|T_g\|_* = \|g\|_{L^q}$.

(c) Let $g \in L^2(\mathbb{R})$. Let $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be defined by

$$f_n(x) = \frac{x}{n} \cdot \mathbb{1}_{[0,1]}(x).$$

Prove that $(f_n)_{n=1}^{\infty}$ converges in $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$ and hence prove that $(T_g(f_n))_{n=1}^{\infty}$ converges in $\mathbb{R}$.

Q7 (a) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be pre-Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ respectively. In this question, you may use the fact that the space $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle_{\mathcal{H}} = \langle x_1, y_1 \rangle_{\mathcal{H}_1} + \langle x_2, y_2 \rangle_{\mathcal{H}_2}$$

is a pre-Hilbert space. Let $\| \cdot \|_1$, $\| \cdot \|_2$ denote the norms on $\mathcal{H}_1$, $\mathcal{H}_2$ that are derived from the inner products $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ respectively. Prove that if $\mathcal{H}_1$, $\mathcal{H}_2$ are Hilbert spaces with respect to the norms $\| \cdot \|_1$, $\| \cdot \|_2$, then $\mathcal{H}$ is a Hilbert space with respect to the norm $\| \cdot \|_{\mathcal{H}} = \sqrt{\langle \cdot, \cdot \rangle_{\mathcal{H}}}$.

(b) Let $k \in \mathbb{N}$. Let $f : [-\pi, \pi] \to \mathbb{R}$ be a 2π-periodic function such that its derivatives

$$\frac{d^j}{dx^j} f = f^{(j)}(x), \quad 0 \le j \le k,$$

exist and are continuous. Recall that the Fourier coefficients of f are defined as

$$a_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy, \quad n \in \mathbb{Z}.$$

Prove that

$$n^k a_n(f) \to 0 \text{ as } n \to +\infty.$$

Remember to state the names of any results from lectures that you use. [You may use the integration by parts formula

$$\int_a^b u(x) \frac{dv}{dx}(x)\, dx = [u(x)v(x)]_a^b - \int_a^b \frac{du}{dx}(x)\, v(x)\, dx.$$

for differentiable functions $u, v : [a, b] \to \mathbb{R}$, and the fact that the derivative of a 2π-periodic function is 2π-periodic].

SECTION C

Q8 Let $(X, \mathcal{X}, \mu)$ and $(Y, \mathcal{Y}, \nu)$ be measure spaces.

(a) State the Product Measure Theorem.

(b) State Tonelli's Theorem.

(c) How do the assumptions of Fubini's Theorem differ from the assumptions of Tonnelli's Theorem?

(d) Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ be the σ-finite measure space given by the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ and the Lebesgue measure μ. Let π be the measure given in the Product Measure Theorem for $(X, \mathcal{X}, \mu) = (Y, \mathcal{Y}, \nu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$. Consider the measurable function $\psi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\psi(x, y) = \begin{cases} x^2 y, & -1 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Can Fubini's Theorem be applied to evaluate $\int_{\mathbb{R}^2} \psi \, d\pi$? Justify your response. [You may use the fact that if $\int_{\mathbb{R}^2} |\psi| \, d\pi < \infty$ then ψ is integrable].