Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH41320-WE01

Title: Riemannian Geometry V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 Prove or disprove the following assertions:

(a) *The subset of* $\mathbb{R}^3$ *given by*

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0\}$$

is a smooth submanifold of $\mathbb{R}^3$.

(b) *The subset of* $\mathbb{R}^3$ *given by*

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 2y^2 + 7z^2 = 1\}$$

is a smooth 2*-dimensional submanifold of* $\mathbb{R}^3$.

Q2 (a) State the definition of a differentiable vector field on a smooth manifold M.

(b) Let

$$X = \frac{\partial}{\partial x} \text{ and } Y = x\frac{\partial}{\partial y}$$

be two differentiable vector fields on $\mathbb{R}^2$ equipped with the usual chart given by the identity map. Show that XY is not a linear derivation on $C^\infty(\mathbb{R}^2)$ by applying it to the product of the smooth real-valued functions f, g on $\mathbb{R}^2$ given by $f(x, y) = x$ and $g(x, y) = y$.

(c) Compute the Lie bracket of the vector fields X and Y.

Q3 Prove or disprove the following assertions:

(a) *Let*

$$P = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } (x, y, z) \neq (0, 0, 1)\} \subset \mathbb{R}^3.$$

Then the Riemannian metric induced on P *by the Euclidean metric on* $\mathbb{R}^3$ *is geodesically complete.*

(b) *There exists a Riemannian metric* g *on* $\mathbb{R}^2$ *such that the Christoffel symbols* $\Gamma_{ij}^k(p)$ *of its Levi–Civita connection* ∇ *in local coordinates at some point* p *are given by*

$$\begin{aligned}
\Gamma_{11}^1(p) &= 0, & \Gamma_{11}^2(p) &= 0,\\
\Gamma_{12}^1(p) &= 1, & \Gamma_{12}^2(p) &= 0,\\
\Gamma_{21}^1(p) &= -1, & \Gamma_{21}^2(p) &= 0,\\
\Gamma_{22}^1(p) &= 1, & \Gamma_{22}^2(p) &= 0.
\end{aligned}$$

(c) *Every complete Riemannian manifold with positive sectional curvature is compact.*

Q4 Let $P = \{(x, y, z) \mid z = x^2 + y^2\} \subset \mathbb{R}^3$ be a paraboloid and let $\varphi\colon P \to \mathbb{R}^2$ be a global chart given by

$$\varphi^{-1}(x, y) = (x, y, x^2 + y^2).$$

Consider the smooth map $f\colon \mathbb{R}^2 \to P$ given by

$$f(u, v) = (u, 3v, u^2 + 9v^2).$$

(a) Compute the matrix of the differential $Df_{(u_o,v_o)}$ at a point $(u_o, v_o) \in \mathbb{R}^2$ in terms of the bases of $T_{(u_o,v_o)}\mathbb{R}^2$ and $T_{f(u_o,v_o)}P$ determined by the coordinate tangent vectors.

(b) Find a Riemannian metric g on P such that f becomes an isometry with respect to the Euclidean metric on $\mathbb{R}^2$ and compute its matrix expression (g_{ij}) in the coordinates (x, y). You may assume, without proof, that f is a diffeomorphism.

SECTION B

Q5 (a) Let $\varphi^{-1}(x, y) = (f(x)\cos(y), f(x)\sin(y), h(x))$ with $f\colon [a, b] \to (0, \infty)$ be a coordinate chart of a smooth surface of revolution $S \subset \mathbb{R}^3$. Assume that S carries the Riemannian metric g induced by the Euclidean metric on $\mathbb{R}^3$. Assume, furthermore, that

$$(f'(x))^2 + (h'(x))^2 = 1 \qquad \text{for all } x \in [a, b].$$

Compute the local expression (g_{ij}) of the Riemannian metric g in the coordinates given by the chart φ and show that

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0,$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = \frac{f'(x)}{f(x)} \frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -f(x)f'(x) \frac{\partial}{\partial x}.$$

(b) Prove or disprove the following statement: *Any two tori T^m and T^n with $m \neq n$ are locally diffeomorphic.*

Q6 Fix $n \geq 2$ and let $M(n,\mathbb{R})$ denote the set of $n \times n$ matrices with real entries, $\mathrm{Sym}(n) = \{A \in M(n,\mathbb{R}) \mid A^T = A\}$ be the set of $n \times n$ real symmetric matrices, and

$$J = \begin{pmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & -1 \end{pmatrix},$$

where Id_k denotes the $k \times k$ identity matrix. Let

$$\mathrm{O}(n-1,1) = \{A \in M(n,\mathbb{R}) \mid AJA^T = J\}.$$

(a) Show that each matrix in $\mathrm{O}(n-1,1)$ is invertible and that $\mathrm{O}(n-1,1)$ is a group under matrix multiplication.

(b) Define $f\colon M(n,\mathbb{R}) \to \mathrm{Sym}(n)$ by $f(A) = AJA^T$. Show that

$$Df_A(B) = AJB^T + BJA^T$$

for any $A, B \in M(n,\mathbb{R})$ and conclude that, for $A \in \mathrm{O}(n-1,1)$ and $C \in \mathrm{Sym}(n)$,

$$Df_A\left(\frac{1}{2}CJA\right) = C.$$

(c) Show that $J \in \mathrm{Sym}(n)$ is a regular value of f.

(d) Show that $\mathrm{O}(n-1,1)$ is a Lie group and determine its dimension.

Q7 (a) Let $M = S^2(1) \times S^2(1)$ be the Riemannian product of two copies of the 2-dimensional unit round sphere. Show that M has non-negative sectional curvature. You may use, without proof, results on product Riemannian manifolds seen in the lecture.

(b) Does there exist a geodesic $\gamma\colon [a,b] \to \mathbb{H}^n$ in n-dimensional hyperbolic space and a non-zero Jacobi field J along γ with $J(a) = J(b) = 0$? Justify your answer. You may use, without proof, results on Jacobi fields and curvature seen in the lecture.

(c) Let M be a smooth manifold and let ∇ be an affine connection on M. The *conjugate connection* ∇^* is defined by

$$\nabla^*_X Y = \nabla_X Y - T(X,Y),$$

where $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ is the torsion of ∇. Show that ∇^* is an affine connection on M and that the torsion of ∇^* is $-T$.

Q8 (a) Let (M, g) be a Riemannian manifold and $c\colon [a, b] \to M$ be a geodesic. Let X be a smooth parallel vector field along c. Show that the map $t \mapsto \langle X(t), c'(t)\rangle$ is constant.

(b) State the symmetries of the curvature tensor R of a Riemannian manifold (M, g).

(c) Let (M, g) be an n-dimensional Riemannian manifold and let R be its Riemannian curvature tensor. The *Ricci tensor* of M is defined as

$$\mathrm{Ric}(X, Y) = \mathrm{Tr}(Z \mapsto R(Z, X)Y),$$

where X, Y, Z are smooth vector fields on M, i.e. Ric is the trace of the endomorphism $Z \mapsto R(Z, X)Y$. In terms of an orthonormal basis $\{e_i\}_{i=1}^n$ of the tangent space, Ric is given by

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i).$$

Show that Ric defines a $\binom{2}{0}$ symmetric tensor field on M.