

EXAMINATION PAPER

Examination Session: May/June

2025

Year:

Exam Code:

MATH41320-WE01

Title:

Riemannian Geometry V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Materials i crimited.		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions.	
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.	
	Write your answer in the white-covered answer booklet with barcodes.	
	Begin your answer to each question on a new page.	

Revision:

SECTION A

- Q1 Prove or disprove the following assertions:
 - (a) The subset of \mathbb{R}^3 given by

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0\}$$

is a smooth submanifold of \mathbb{R}^3 .

(b) The subset of \mathbb{R}^3 given by

$$M = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + 2y^2 + 7z^2 = 1 \}$$

is a smooth 2-dimensional submanifold of \mathbb{R}^3 .

- **Q2** (a) State the definition of a differentiable vector field on a smooth manifold M.
 - (b) Let

$$X = \frac{\partial}{\partial x}$$
 and $Y = x \frac{\partial}{\partial y}$

be two differentiable vector fields on \mathbb{R}^2 equipped with the usual chart given by the identity map. Show that XY is not a linear derivation on $C^{\infty}(\mathbb{R}^2)$ by applying it to the product of the smooth real-valued functions f, g on \mathbb{R}^2 given by f(x, y) = x and g(x, y) = y.

- (c) Compute the Lie bracket of the vector fields X and Y.
- Q3 Prove or disprove the following assertions:
 - (a) Let

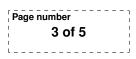
$$P = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } (x, y, z) \neq (0, 0, 1) \} \subset \mathbb{R}^3.$$

Then the Riemannian metric induced on P by the Euclidean metric on \mathbb{R}^3 is geodesically complete.

(b) There exists a Riemannian metric g on \mathbb{R}^2 such that the Christoffel symbols $\Gamma_{ij}^k(p)$ of its Levi-Civita connection ∇ in local coordinates at some point p are given by

$$\begin{split} \Gamma^{1}_{11}(p) &= 0, & \Gamma^{2}_{11}(p) = 0, \\ \Gamma^{1}_{12}(p) &= 1, & \Gamma^{2}_{12}(p) = 0, \\ \Gamma^{1}_{21}(p) &= -1, & \Gamma^{2}_{21}(p) = 0, \\ \Gamma^{1}_{22}(p) &= 1, & \Gamma^{2}_{22}(p) = 0. \end{split}$$

(c) Every complete Riemannian manifold with positive sectional curvature is compact.





Q4 Let $P = \{(x, y, z) \mid z = x^2 + y^2\} \subset \mathbb{R}^3$ be a paraboloid and let $\varphi \colon P \to \mathbb{R}^2$ be a global chart given by

$$\varphi^{-1}(x,y) = (x,y,x^2 + y^2).$$

Consider the smooth map $f \colon \mathbb{R}^2 \to P$ given by

$$f(u, v) = (u, 3v, u^2 + 9v^2).$$

- (a) Compute the matrix of the differential $Df_{(u_o,v_o)}$ at a point $(u_o,v_o) \in \mathbb{R}^2$ in terms of the bases of $T_{(u_o,v_o)}\mathbb{R}^2$ and $T_{f(u_o,v_o)}P$ determined by the coordinate tangent vectors.
- (b) Find a Riemannian metric g on P such that f becomes an isometry with respect to the Euclidean metric on \mathbb{R}^2 and compute its matrix expression (g_{ij}) in the coordinates (x, y). You may assume, without proof, that f is a diffeomorphism.

SECTION B

Q5 (a) Let $\varphi^{-1}(x, y) = (f(x) \cos(y), f(x) \sin(y), h(x))$ with $f: [a, b] \to (0, \infty)$ be a coordinate chart of a smooth surface of revolution $S \subset \mathbb{R}^3$. Assume that S carries the Riemannian metric g induced by the Euclidean metric on \mathbb{R}^3 . Assume, furthermore, that

$$(f'(x))^2 + (h'(x))^2 = 1$$
 for all $x \in [a, b]$.

Compute the local expression (g_{ij}) of the Riemannian metric g in the coordinates given by the chart φ and show that

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0,$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = \frac{f'(x)}{f(x)} \frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -f(x)f'(x)\frac{\partial}{\partial x}.$$

(b) Prove or disprove the following statement: Any two tori T^m and T^n with $m \neq n$ are locally diffeomorphic.

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- **Q6** Fix $n \ge 2$ and let $M(n, \mathbb{R})$ denote the set of $n \times n$ matrices with real entries, $\operatorname{Sym}(n) = \{A \in M(n, \mathbb{R}) \mid A^T = A\}$ be the set of $n \times n$ real symmetric matrices, and

$$J = \begin{pmatrix} \mathrm{Id}_{n-1} & 0\\ 0 & -1 \end{pmatrix},$$

where Id_k denotes the $k \times k$ identity matrix. Let

$$O(n-1,1) = \{A \in M(n,\mathbb{R}) \mid AJA^T = J\}.$$

- (a) Show that each matrix in O(n-1,1) is invertible and that O(n-1,1) is a group under matrix multiplication.
- (b) Define $f: M(n, \mathbb{R}) \to \text{Sym}(n)$ by $f(A) = AJA^T$. Show that

$$Df_A(B) = AJB^T + BJA^T$$

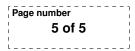
for any $A, B \in M(n, \mathbb{R})$ and conclude that, for $A \in O(n-1, 1)$ and $C \in Sym(n)$,

$$Df_A\left(\frac{1}{2}CJA\right) = C.$$

- (c) Show that $J \in \text{Sym}(n)$ is a regular value of f.
- (d) Show that O(n-1,1) is a Lie group and determine its dimension.
- Q7 (a) Let $M = S^2(1) \times S^2(1)$ be the Riemannian product of two copies of the 2dimensional unit round sphere. Show that M has non-negative sectional curvature. You may use, without proof, results on product Riemannian manifolds seen in the lecture.
 - (b) Does there exist a geodesic $\gamma : [a, b] \to \mathbb{H}^n$ in *n*-dimensional hyperbolic space and a non-zero Jacobi field J along γ with J(a) = J(b) = 0? Justify your answer. You may use, without proof, results on Jacobi fields and curvature seen in the lecture.
 - (c) Let M be a smooth manifold and let ∇ be an affine connection on M. The conjugate connection ∇^* is defined by

$$\nabla_X^* Y = \nabla_X Y - T(X, Y),$$

where $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ is the torsion of ∇ . Show that ∇^* is an affine connection on M and that the torsion of ∇^* is -T.



- **Q8** (a) Let (M, g) be a Riemannian manifold and $c: [a, b] \to M$ be a geodesic. Let X be a smooth parallel vector field along c. Show that the map $t \mapsto \langle X(t), c'(t) \rangle$ is constant.
 - (b) State the symmetries of the curvature tensor R of a Riemannian manifold (M, g).
 - (c) Let (M, g) be an *n*-dimensional Riemannian manifold and let R be its Riemannian curvature tensor. The *Ricci tensor* of M is defined as

$$\operatorname{Ric}(X,Y) = \operatorname{Tr}(Z \mapsto R(Z,X)Y),$$

where X, Y, Z are smooth vector fields on M, i.e. Ric is the trace of the endomorphism $Z \mapsto R(Z, X)Y$. In terms of an orthonormal basis $\{e_i\}_{i=1}^n$ of the tangent space, Ric is given by

$$\operatorname{Ric}(X,Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i).$$

Show that Ric defines a $\binom{2}{0}$ symmetric tensor field on M.