Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH41420-WE01

Title:
Solitons V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 Write down the ball and box rule for one time-step $t \to t+1$ in the single-colour ball and box model. If at time $t = 0$ there are balls in boxes 1, 2, 3 and 8, find the locations of the balls at times $t = 1$, $t = 2$ and $t = 3$, and give the phase shifts undergone by the two solitons during this process.

Q2 (a) Solve the Marchenko equation

$$K(x,z) + F(x{+}z) + \int_{-\infty}^{x} dy\, K(x,y)\, F(y{+}z) = 0$$

to determine the unknown function $K(x,z)$, given that $F(x) = c\exp(cx)\theta(x)$, where c is a constant and

$$\theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0\ . \end{cases}$$

(b) Calculate

$$V(x) = 2\frac{d}{dx}K(x,x)\ .$$

Q3 The Poisson bracket (at time t) of two functionals $F[u]$, $G[u]$ of $u(x,t)$ is defined as

$$\{F[u], G[u]\} := \int_{-\infty}^{+\infty} dx \; \frac{\delta F[u]}{\delta u(x,t)} \frac{\partial}{\partial x} \frac{\delta G[u]}{\delta u(x,t)} \ ,$$

and the time evolution of a functional $F[u]$ is governed by the equation

$$\frac{d}{dt} F[u] = \{H[u], F[u]\}$$

where $H[u]$ is the Hamiltonian. In the following you may assume that fields fall off sufficiently fast as $x \to \pm\infty$ that all boundary terms vanish.

(a) Derive an expression for $\frac{\partial}{\partial t} u(x,t)$, by viewing $u(x,t)$ as

$$u(x,t) = \int_{-\infty}^{+\infty} dy \; \delta(y-x) u(y,t) \ .$$

(b) Find the PDE that governs the time evolution of $u(x,t)$ if

$$H[u] = \int_{-\infty}^{+\infty} dx \; \left[u(x,t)^3 - \frac{1}{2} u_x(x,t)^2 \right] \ .$$

(c) Use the Poisson bracket and the Hamiltonian $H[u]$ given in the previous part to calculate the time derivative of

$$Q[u] = \int_{-\infty}^{+\infty} dx \; u(x,t)^2 \ .$$

SECTION B

Q4 (a) Suppose that the equation of motion for a field $u(x,t)$ is such that

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

for a pair of densities $\rho(u, u_t, u_x, \dots)$ and $j(u, u_t, u_x, \dots)$. Show that the charge $Q = \int_{-\infty}^{\infty} \rho \, dx$ is conserved, provided the boundary conditions for u imply a certain condition on the limits of j which you should state.

(b) Find two independent conserved charges for the equation

$$u_t + 20u^3 u_x + u_{xxx} = 0$$

with boundary conditions u, u_x, $u_{xx} \to 0$ as $x \to \pm\infty$.

Q5 A field $u(x,t)$, defined on the infinite line $-\infty < x < \infty$, has energy

$$E[u] = \int_{-\infty}^{\infty} \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u^4(u^2-1)^2 \, dx \, .$$

(a) If u is to have finite energy, what values can the topological charge

$$Q_0[u] = \int_{-\infty}^{\infty} \tfrac{\partial}{\partial x} u \, dx = [\, u \,]_{x=-\infty}^{x=+\infty}$$

take? For each case, give the limits of u at $x = \pm\infty$ which lead to that charge.

(b) Now suppose $Q_0[u] = Q^{(i)}$, where $Q^{(i)}$ is one of the possible values of the topological charge that you identified in part (a), and suppose in addition that $Q^{(i)} > 0$. Show that $E[u] \geq K^{(i)}$, where $K^{(i)}$ is a constant which you should find for each $Q^{(i)} > 0$.

(c) Write down the conditions for the bounds you found in part (b) to be saturated, so that $E[u] = K^{(i)}$. Is it possible for these to be satisfied if $Q^{(i)}$ is the largest of the values that you found in part (a)? (You can assume that solutions *do* exist which saturate the bound for all other possible positive values of $Q^{(i)}$.)

Q6 The time-independent Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x)$$

for the potential $V(x) = -2\,\text{sech}^2(x)$ has the general solution

$$\psi(x) = Ae^{ikx}\left(-ik + \tanh(x)\right) + Be^{-ikx}\left(ik + \tanh(x)\right)\ ,$$

where A and B are constants and $k^2 > 0$.

(a) Find the reflection and transmission coefficients $R(k)$ and $T(k)$ for the potential

$$V(x) = \begin{cases} 0\ , & x < 0 \\ -2\,\text{sech}^2(x)\ , & x \geq 0\ . \end{cases}$$

(b) Find all (unnormalised) bound state solutions for the potential in part (a).

Q7 The pair of operators

$$\begin{aligned} L &= (\phi_1 + i\phi_2) - 2iz\ \phi_3 + z^2(\phi_1 - i\phi_2) \\ M &= i\phi_3 - z\ (\phi_1 - i\phi_2) \end{aligned}$$

satisfies the Lax equation $\dot{L} = [M, L]$ for all values of the parameter z, where the dot denotes a time derivative, and ϕ_1, ϕ_2, ϕ_3 do not depend on z.

(a) Find time evolution equations for ϕ_1, ϕ_2, ϕ_3.

(b) Assume that $\phi_n = iw_n\sigma_n/2$ (indices $n = 1, 2, 3$ are not summed over), where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ .$$

Find time evolution equations for the functions w_1, w_2, w_3, and conservation laws for that time evolution.

SECTION C

Q8 Consider the sine-Gordon model on a half-infinite line, $-\infty < x \leq 0$. For $x < 0$ the field $u(x,t)$ solves the bulk sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0\,.$$

As $x \to -\infty$ the boundary conditions are u_t, u_x, $1 - \cos u \to 0$, while at $x = 0$ a Dirichlet boundary condition is imposed:

$$u(0,t) = 0\,.$$

(a) Prove that the half-line energy

$$E[u] = \int_{-\infty}^{0} \mathcal{E}\, dx$$

is conserved, where $\mathcal{E}$ is the (bulk) sine-Gordon energy density:

$$\mathcal{E} = \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + (1 - \cos u)\,.$$

(Hint/reminder: begin by showing that for $x < 0$, $\partial\mathcal{E}/\partial t + \partial j/\partial x = 0$, where j is some other density, which you should determine.)

(b) Explain briefly how you would use the method of images, and a particular two-soliton solution on the full line, to find an exact solution for the scattering of a single sine-Gordon kink off this boundary.