Durham University

EXAMINATION PAPER

Examination Session: May/June	**Year:** 2025	**Exam Code:** MATH41520-WE01

Title: Topics in Algebra and Geometry V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 (a) Show that $\tau_0 := -\frac{1}{2} + \frac{\sqrt{3}i}{6} \in \mathbb{H}$ is stabilized by $\gamma_0 := \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$ under linear fractional transformations.

(b) Show that for the automorphy factor we have $j(\gamma_0, \tau_0) = -e^{2\pi i/6}$. Conclude that any modular form for $\Gamma_0(3)$ of weight k not divisible by 6 vanishes at τ_0.

Q2 (a) Define Dedekind's eta-function $\eta(\tau)$ and state (without proof) the transformation formulas under the generators S and T of $\mathrm{SL}_2(\mathbb{Z})$.

(b) Show that $Im(\tau)^{1/4}|\eta(\tau)|$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$ and bounded on the entire upper half plane $\mathbb{H}$.

Q3 Let f, g and h be non-zero arithmetic functions.

(a) Prove that if f is a multiplicative function, then $f(1) = 1$.

(b) Define the Dirichlet convolution of two arithmetic functions f and g.

(c) Show that if f and g are multiplicative, then $h = f * g$ is also multiplicative.

(d) Suppose $h = f * g$. Let $p \neq q$ be distinct primes. Show that if h and f are multiplicative, then $g(pq) = g(p)g(q)$.

Q4 Recall that $\Lambda(n)$ denotes the von Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ is a prime power,} \\ 0 & \text{otherwise.} \end{cases}$$

(a) State the Abel summation formula, also known as the partial summation formula.

(b) Show

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

You may use here that

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

(c) Apply partial summation to deduce that

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + O(1), \qquad \text{for } x \geq 2.$$

SECTION B

Q5 Consider the Hecke subgroup $\Gamma_0(3) \subset \mathrm{SL}_2(\mathbb{Z})$. For f a non-zero meromorphic form of weight k for $\Gamma_0(3)$, the valence formula reads

$$ord_\infty(f) + ord_0(f) + \frac{1}{3}ord_{\tau_0}(f) + \sum_{P\in\mathcal{F}';P\neq\tau_0} ord_P(f) = \frac{k}{3}.$$

Here $\mathcal{F}'$ denotes a set of representatives for the $\Gamma_0(3)$-action on $\mathbb{H}$ and - see Q1 of this exam - $\tau_0 \in \mathbb{H}$ is the only point (up to equivalence) with non-trivial stabilizer γ_0 of order 3 in $\Gamma_0(3)/\pm I$. Further, $\Gamma_0(3)$ has two equivalence classes of cusps, ∞ and 0.

(a) Describe the zeros of a (non-zero) modular form in the spaces $M_2(\Gamma_0(3))$, $S_6(\Gamma_0(3))$, and $S_8(\Gamma_0(3))$.

(b) Show that a non-zero cusp form in $S_6(\Gamma_0(3))$ cannot be expressed as a polynomial of forms for $\Gamma_0(3)$ of weight 2 and 4.

(c) The Eisenstein series $F(\tau) := E_4(\tau)$ and $G(\tau) := E_4(3\tau)$ form a basis of $M_4(\Gamma_0(3))$. Use these forms to explicitly construct a non-zero cusp form f in $S_8(\Gamma_0(3))$. You may assume that G takes the value $c = 3^{-4}$ at the cusp 0.

(d) You may assume that $M_2(\Gamma_0(3))$ is one-dimensional spanned by a form $E(\tau)$. Use this and the form g constructed in the previous part to explicitly construct a cusp form of weight 6 for $\Gamma_0(3)$.

Q6 Let $Q_1(x, y) = x^2 + 5y^2$ and $Q_2(x, y) = 2x^2 + 2xy + 3y^2 = 2(x + y/2)^2 + \frac{5}{2}y^2$ be two binary positive definite quadratic forms of level 20.

You may assume that the associated theta series $\theta(\tau, Q_i)$ are modular forms of weight 1 for $\Gamma_0(20)$ for the quadratic residue (Dirichlet) character $\chi_{-20}(n)$ given by $\chi_{-20}(n) = (-1)^{(n-1)/2}\left(\frac{n}{5}\right)$ for $gcd(n, 20) = 1$ and 0 else (in particular $\chi_{-20}(2) = 0$). So $\theta(\gamma\tau, Q_i) = \chi_{-20}(d)j(\gamma, \tau)\theta(\tau, Q_i)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(20)$. You also may assume that the space of such forms is two-dimensional.

The theory of Hecke operators extends to this space with T_p given by "$b_n = a_{pn} + \chi_{-20}(p)a_{n/p}$" for all primes (including 2 and 5).

(a) Compute the Fourier coefficients for both theta series up to index 3.

(b) Compute T_2 for both $\theta(\tau, Q_1)$ and $\theta(\tau, Q_2)$ and find eigenvectors f_1 and f_2 and their eigenvalues under T_2. Conclude that f_1 and f_2 are eigenvectors for all Hecke operators.

(c) One of the forms you have found, say $f_1(\tau) = \sum_{n=0}^{\infty} a_n q^n$, has non-zero constant coefficient a_0. Show that, if normalized ($a_1 = 1$), we have

$$a_p = \chi_{-20}(p) + 1$$

for all primes p. Describe what this means in terms of the representation numbers of the quadratic forms Q_1 and Q_2. Verify this for $p = 2$ and $p = 3$.

Q7 Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, denote the Riemann zeta function, absolutely convergent in the region $Re(s) > 1$.

(a) Prove that $\sum_{k=2}^{\infty}(\zeta(k) - 1) = 1$.
(Hint: You may use the Dirichlet series here.)

(b) Show that for any $\sigma > 1$ and any $t \in \mathbb{R}$,

$$\log \zeta(\sigma + it) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma} \log(n)}(\cos(\log(n)t) - i \sin(\log(n)t)).$$

Here $\Lambda(n)$ denotes the von Mangoldt function as defined in Question 4.

(c) Prove that for any $\sigma > 1$ and any $t \in \mathbb{R}$,

$$|\zeta(\sigma)^5 \zeta(\sigma + it)^8 \zeta(\sigma + 2it))^4 \zeta(\sigma + 3it)| \geq 1.$$

(Hint: You may use here the relations $2\cos^2\theta = 1+\cos 2\theta$ and $2\cos 2\theta \cos\theta = \cos\theta + \cos 3\theta$.)

(d) Deduce that $\zeta(1 + it) \neq 0$ for any $t \in \mathbb{R}$. You may use here that ζ has only one simple pole at $s = 1$.

Q8 Throughout this question, let q be a positive integer, let $d \mid q$ satisfying $d < q$ be a proper divisor, and let χ be a non-trivial Dirichlet character modulo q. Given any real number $N \geq 1$, let

$$S_{\chi}(N) := \sum_{1 \leq a \leq N} \chi(a). \tag{1}$$

(a) Prove that

$$\sum_{1 \leq a \leq q} \chi(a) = 0.$$

(b) Suppose you are given that there exists an integer c satisfying $c \equiv 1 \bmod d$, $\gcd(c, q) = 1$ and $\chi(c) \neq 1$. Then prove that we must have

$$\sum_{\substack{1 \leq a \leq q \\ a \equiv 1 \bmod d}} \chi(a) = 0.$$

(c) Using partial summation or otherwise, prove that for $Re(s) > 1$,

$$L(s, \chi) = s \int_1^{\infty} y^{-1-s} S_{\chi}(y) dy,$$

where $S_{\chi}(y)$ be as in (1) and $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ denotes the Dirichlet L-function. Show further that the above equation gives an analytic continuation of $L(s, \chi)$ to the region $Re(s) > 0$.

(d) Suppose you are given that $|S_{\chi}(N)| \ll \sqrt{q} \log q$, for every $N \geq 1$. Then prove that for any $s = \sigma + it$ with $1/2 < \sigma < 1$,

$$|L(s, \chi)| \ll |s| q^{\frac{1-\sigma}{2}} \log q.$$

Here, the implied constant is allowed to depend on σ.

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