

## **EXAMINATION PAPER**

Examination Session: May/June

Title:

Year: 2025

Exam Code:

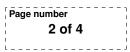
MATH41520-WE01

Topics in Algebra and Geometry V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions.	
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.	
	Write your answer in the white-covered answer booklet with barcodes.	
	Begin your answer to each question on a new page.	

Revision:



## SECTION A

- **Q1** (a) Show that  $\tau_0 := -\frac{1}{2} + \frac{\sqrt{3}i}{6} \in \mathbb{H}$  is stabilized by  $\gamma_0 := \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$  under linear fractional transformations.
  - (b) Show that for the automorphy factor we have  $j(\gamma_0, \tau_0) = -e^{2\pi i/6}$ . Conclude that any modular form for  $\Gamma_0(3)$  of weight *k* not divisible by 6 vanishes at  $\tau_0$ .
- **Q2** (a) Define Dedekind's eta-function  $\eta(\tau)$  and state (without proof) the transformation formulas under the generators *S* and *T* of SL<sub>2</sub>( $\mathbb{Z}$ ).
  - (b) Show that  $Im(\tau)^{1/4}|\eta(\tau)|$  is invariant under  $SL_2(\mathbb{Z})$  and bounded on the entire upper half plane  $\mathbb{H}$ .
- **Q3** Let f, g and h be non-zero arithmetic functions.
  - (a) Prove that if *f* is a multiplicative function, then f(1) = 1.
  - (b) Define the Dirichlet convolution of two arithmetic functions f and g.
  - (c) Show that if f and g are multiplicative, then h = f \* g is also multiplicative.
  - (d) Suppose h = f \* g. Let  $p \neq q$  be distinct primes. Show that if *h* and *f* are multiplicative, then g(pq) = g(p)g(q).
- **Q4** Recall that  $\Lambda(n)$  denotes the von Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ is a prime power,} \\ 0 & \text{otherwise.} \end{cases}$$

- (a) State the Abel summation formula, also known as the partial summation formula.
- (b) Show

$$\sum_{p\leq x}\frac{\log p}{p}=\log x+O(1).$$

You may use here that

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

(c) Apply partial summation to deduce that

$$\sum_{p \le x} \frac{1}{p} = \log \log x + O(1), \quad \text{ for } x \ge 2.$$

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## SECTION B

**Q5** Consider the Hecke subgroup  $\Gamma_0(3) \subset SL_2(\mathbb{Z})$ . For *f* a non-zero meromorphic form of weight *k* for  $\Gamma_0(3)$ , the valence formula reads

$$ord_{\infty}(f) + ord_{0}(f) + \frac{1}{3}ord_{\tau_{0}}(f) + \sum_{P \in \mathcal{F}'; P \neq \tau_{0}} ord_{P}(f) = \frac{k}{3}.$$

Here  $\mathcal{F}'$  denotes a set of representatives for the  $\Gamma_0(3)$ -action on  $\mathbb{H}$  and - see Q1 of this exam -  $\tau_0 \in \mathbb{H}$  is the only point (up to equivalence) with non-trivial stabilizer  $\gamma_0$  of order 3 in  $\Gamma_0(3)/\pm I$ . Further,  $\Gamma_0(3)$  has two equivalence classes of cusps,  $\infty$  and 0.

- (a) Describe the zeros of a (non-zero) modular form in the spaces  $M_2(\Gamma_0(3))$ ,  $S_6(\Gamma_0(3))$ , and  $S_8(\Gamma_0(3))$ .
- (b) Show that a non-zero cusp form in  $S_6(\Gamma_0(3))$  cannot be expressed as a polynomial of forms for  $\Gamma_0(3)$  of weight 2 and 4.
- (c) The Eisenstein series  $F(\tau) := E_4(\tau)$  and  $G(\tau) := E_4(3\tau)$  form a basis of  $M_4(\Gamma_0(3))$ . Use these forms to explicitly construct a non-zero cusp form *f* in  $S_8(\Gamma_0(3))$ . You may assume that *G* takes the value  $c = 3^{-4}$  at the cusp 0.
- (d) You may assume that  $M_2(\Gamma_0(3))$  is one-dimensional spanned by a form  $E(\tau)$ . Use this and the form *g* constructed in the previous part to explicitly construct a cusp form of weight 6 for  $\Gamma_0(3)$ .
- **Q6** Let  $Q_1(x, y) = x^2 + 5y^2$  and  $Q_2(x, y) = 2x^2 + 2xy + 3y^2 = 2(x + y/2)^2 + \frac{5}{2}y^2$  be two binary positive definite quadratic forms of level 20.

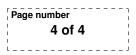
You may assume that the associated theta series  $\theta(\tau, Q_i)$  are modular forms of weight 1 for  $\Gamma_0(20)$  for the quadratic residue (Dirichlet) character  $\chi_{-20}(n)$  given by  $\chi_{-20}(n) = (-1)^{(n-1)/2} \left(\frac{n}{5}\right)$  for gcd(n, 20) = 1 and 0 else (in particular  $\chi_{-20}(2) = 0$ ). So  $\theta(\gamma\tau, Q_i) = \chi_{-20}(d)j(\gamma, \tau)\theta(\tau, Q_i)$  for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(20)$ . You also may assume that the space of such forms is two-dimensional.

The theory of Hecke operators extends to this space with  $T_p$  given by " $b_n = a_{pn} + \chi_{-20}(p)a_{n/p}$ " for all primes (including 2 and 5).

- (a) Compute the Fourier coefficients for both theta series up to index 3.
- (b) Compute  $T_2$  for both  $\theta(\tau, Q_1)$  and  $\theta(\tau, Q_2)$  and find eigenvectors  $f_1$  and  $f_2$  and their eigenvalues under  $T_2$ . Conclude that  $f_1$  and  $f_2$  are eigenvectors for all Hecke operators.
- (c) One of the forms you have found, say  $f_1(\tau) = \sum_{n=0}^{\infty} a_n q^n$ , has non-zero constant coefficient  $a_0$ . Show that, if normalized  $(a_1 = 1)$ , we have

$$a_p = \chi_{-20}(p) + 1$$

for all primes *p*. Describe what this means in terms of the representation numbers of the quadratic forms  $Q_1$  and  $Q_2$ . Verify this for p = 2 and p = 3.



- **Q7** Let  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ , denote the Riemann zeta function, absolutely convergent in the region Re(s) > 1.
  - (a) Prove that  $\sum_{k=2}^{\infty} (\zeta(k) 1) = 1$ . (Hint: You may use the Dirichlet series here.)
  - (b) Show that for any  $\sigma > 1$  and any  $t \in \mathbb{R}$ ,

$$\log \zeta(\sigma + it) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma} \log(n)} (\cos(\log(n)t) - i \sin(\log(n)t)).$$

Here  $\Lambda(n)$  denotes the von Mangoldt function as defined in Question 4.

(c) Prove that for any  $\sigma > 1$  and any  $t \in \mathbb{R}$ ,

$$|\zeta(\sigma)^5\zeta(\sigma+it)^8\zeta(\sigma+2it))^4\zeta(\sigma+3it)| \geq 1.$$

(Hint: You may use here the relations  $2\cos^2 \theta = 1 + \cos 2\theta$  and  $2\cos 2\theta \cos \theta = \cos \theta + \cos 3\theta$ .)

- (d) Deduce that  $\zeta(1+it) \neq 0$  for any  $t \in \mathbb{R}$ . You may use here that  $\zeta$  has only one simple pole at s = 1.
- **Q8** Throughout this question, let *q* be a positive integer, let  $d \mid q$  satisfying d < q be a proper divisor, and let  $\chi$  be a non-trivial Dirichlet character modulo *q*. Given any real number  $N \ge 1$ , let

$$S_{\chi}(N) := \sum_{1 \le a \le N} \chi(a).$$
<sup>(1)</sup>

(a) Prove that

$$\sum_{1\leq a\leq q}\chi(a)=0.$$

(b) Suppose you are given that there exists an integer *c* satisfying  $c \equiv 1 \mod d$ , gcd(c, q) = 1 and  $\chi(c) \neq 1$ . Then prove that we must have

$$\sum_{\substack{1 \le a \le q \\ a \equiv 1 \mod d}} \chi(a) = 0.$$

(c) Using partial summation or otherwise, prove that for Re(s) > 1,

$$L(s,\chi)=s\int_1^\infty y^{-1-s}S_\chi(y)dy,$$

where  $S_{\chi}(y)$  be as in (1) and  $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$  denotes the Dirichlet *L*-function. Show further that the above equation gives an analytic continuation of  $L(s, \chi)$  to the region Re(s) > 0.

(d) Suppose you are given that  $|S_{\chi}(N)| \ll \sqrt{q} \log q$ , for every  $N \ge 1$ . Then prove that for any  $s = \sigma + it$  with  $1/2 < \sigma < 1$ ,

$$|L(s,\chi)| \ll |s|q^{\frac{1-\sigma}{2}}\log q.$$

Here, the implied constant is allowed to depend on  $\sigma$ .