Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH4161-WE01

Title:
Algebraic Topology IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 Triangulate the space $S^1 \vee S^1$ and use your triangulation to compute $H_1^{\text{simp}}(S^1 \vee S^1; \mathbb{Z})$ directly from the definition of simplicial homology, describing the set of all 1-cycles in terms of the simplices of your triangulation.

Q2 There is a short exact sequence of chain complexes where all the chain groups are free $\mathbb{Z}$-modules

$$0 \to C_* \xrightarrow{\alpha} D_* \xrightarrow{\beta} E_* \to 0 \,.$$

Part of this is as follows

$$\begin{array}{ccccc}
0 \to 0 & \xrightarrow{\alpha_{n+1}} & 0 & \xrightarrow{\beta_{n+1}} & 0 \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \mathbb{Z} & \xrightarrow{\alpha_n} & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\beta_n} & \mathbb{Z} \oplus \mathbb{Z} \to 0 \\
\partial_n^C \downarrow & & \partial_n^D \downarrow & & \partial_n^E \downarrow \\
0 \to \mathbb{Z} & \xrightarrow{\alpha_{n-1}} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\beta_{n-1}} & \mathbb{Z} \to 0 \\
\partial_{n-1}^C \downarrow & & \partial_{n-1}^D \downarrow & & \partial_{n-1}^E \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

You are told that

$$\begin{array}{lcr}
\partial_n^C(x) = 2x & \partial_n^D(x,y,z) = (x-2y, x-2y) & \partial_n^E(x,y) = (0,0) \\
\beta_n(x,y,z) = (x,z) & \text{and} & \alpha_{n-1}(x) = (x,x)\,.
\end{array}$$

Compute $H_n(E)$ and $H_{n-1}(C)$ and the homomorphism $\Delta\colon H_n(E) \to H_{n-1}(C)$ in the long exact sequence given by the Snake Lemma.

Q3 (a) Give the definition of the real and complex projective spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$.

(b) Prove that the real projective space $\mathbb{RP}^1$ is homeomorphic to the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

(c) In class, we inductively proved that $\mathbb{CP}^n$ has a CW-structure of exactly one $2k$-cell for each $k = 0, \dots, n$ by providing homeomorphisms

$$f\colon \mathbb{CP}^{n-1} \cup_\varphi D^{2n} \to \mathbb{CP}^n,$$

where $\varphi\colon S^{2n-1} = \partial D^{2n} \to \mathbb{CP}^{n-1}$ is the attaching map of the $2n$-cell D^{2n}. Write down the attaching map φ, and the map f. Prove that the map f is surjective. (You do not need to prove that f is injective, continuous, or a homeomorphism.)

(d) Specify the cohomology groups $H^k(\mathbb{CP}^n; \mathbb{Z})$ for all $k \geq 0$, and justify your answer. Describe the ring $H^*(\mathbb{CP}^n; \mathbb{Z})$ with the cup-product as the multiplication, but without proving your claim.

Q4 Consider the sequence of abelian groups

$$0 \to C_3 = \mathbb{Z}/15 \xrightarrow{\cdot 6} C_2 = \mathbb{Z}/15 \xrightarrow{\cdot 10} C_1 = \mathbb{Z}/15 \xrightarrow{0} C_0 = \mathbb{Z} \to 0,$$

where $\cdot 6\colon \mathbb{Z}/15 \to \mathbb{Z}/15$ denotes the map induced from multiplication with 6 on $\mathbb{Z}$, and likewise $\cdot 10\colon \mathbb{Z}/15 \to \mathbb{Z}/15$ denotes the map induced from multiplication with 10 on $\mathbb{Z}$.

(a) Show that this sequence is a complex of abelian groups (C_*, ∂_*).

(b) Determine its homology groups $H_*(C_*)$, and provide the orders of the group $H_k(C_*)$ for each k.

(c) Determine the cohomology groups $H^*(C^*; \mathbb{Z}/5)$ of the dual complex with coefficients in $\mathbb{Z}/5$, and provide the orders of the cohomology groups $H^k(C^*; \mathbb{Z}/5)$ for each k.

SECTION B

Q5 A space X is constructed by removing the interior of a small open disc from a torus, and gluing in a Möbius band M by identifying the boundary of M to the boundary circle of the punctured torus. Use the Mayer Vietoris sequence to compute $H_n(X; \mathbb{Z}/2)$ and $H_n(X; \mathbb{Z}/3)$. You may use the fact proved in lectures that the punctured torus is homotopy equivalent to $S^1 \vee S^1$.

Q6 State whether you think the following statements are true or false. Prove those statements you think are true, and give a counter-example (with brief justification) for those you think are false.

(a) Any short exact sequence of abelian groups $0 \to \mathbb{Z} \to G \to H \to 0$ splits, (i.e., $G \cong H \oplus \mathbb{Z}$).

(b) Suppose $E_*(-)$ is a reduced homology theory and $\mathbf{p}$ is the one point space. Then the fact that $E_n(\mathbf{p}) = 0$ for all n can be deduced using just the first three Eilenberg-Steenrod axioms (i.e., functoriality, homotopy and the long exact sequence axioms, the excision axiom is not needed).

(c) If A is a retract of X then $H_n(X) = H_n(A) \oplus H_n(X, A)$.

Q7 In this question, we suppose we are given a CW-complex X with cellular chain groups $C_*(X)$ given by

$$\begin{aligned} C_0(X) &\cong \mathbb{Z}, \\ C_1(X) &\cong \mathbb{Z}^4, \\ C_2(X) &\cong \mathbb{Z}, \\ C_3(X) &= 0, \\ C_4(X) &= 0, \\ C_5(X) &\cong \mathbb{Z}, \\ C_6(X) &\cong \mathbb{Z} \end{aligned}$$

and all other chain groups equal to zero. We do not specify, at this stage, what the boundary maps $\partial_k \colon C_k(X) \to C_{k-1}(X)$ are supposed to be.

(a) Recall the statement of Poincaré duality.

(b) Estimate what the ranks $b_k(X) = \operatorname{rank}(H_k(X;\mathbb{Z}))$ of the homology groups $H_k(X;\mathbb{Z})$ could possibly be as a result of this complex.

(c) Show that the CW-complex X cannot be homotopy equivalent to a closed orientable manifold of any dimension $n \geq 3$. Can it be homotopy equivalent to a closed orientable manifold of dimension $n = 2$, and if so, to which?

(d) Describe a CW-complex such that the above cellular chain complex results where the boundary maps ∂_k are equal to zero for all k.

(e) By drawing pictures, sketch a cell decomposition of the 2-dimensional sphere where the number of 0-cells is different from the number of 2-cells.

Q8 In this problem, W will denote a closed, connected, oriented 4-dimensional manifold.

(a) State the Universal Coefficient Theorem.

(b) Show that for any topological space X the cohomology group $H^1(X;\mathbb{Z})$ never contains a non-zero torsion subgroup.

(c) Show that $H_3(W;\mathbb{Z})$ has zero torsion subgroup.

(d) Show that $H_1(W;\mathbb{Z})$ and $H_2(W;\mathbb{Z})$ contain isomorphic torsion subgroups.

(e) If we denote by $b_i(W) = \mathrm{rank}(H_i(W;\mathbb{Z}))$, show that the Euler characteristic satisfies

$$\chi(W) = 2 - 2b_1(W) + b_2(W).$$

Here the rank of a finitely generated abelian group is defined to be the rank of a maximal free abelian subgroup.

(f) Determine the Euler characteristic of the 4-manifolds $S^2 \times S^2$ and of $\mathbb{CP}^2$. You do not need to prove your claims.

(g) Provide an example of a closed, orientable, 4-dimensional manifold W with Euler characteristic zero, $\chi(W) = 0$. You do not need to give a proof of your claim.