Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH41920-WE01

Title:
Geometry V

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 (a) Let $R_{A,\pi/4}, R_{B,\pi/4} : \mathbb{E}^2 \to \mathbb{E}^2$ be rotations by $\pi/4$ in counterclockwise direction in the Euclidean plane about $A \in \mathbb{E}^2$ and $B \in \mathbb{E}^2$, $A \neq B$, respectively. Denote $\varphi = R_{B,\pi/4} \circ R_{A,\pi/4}$. Find the type of φ.

(b) Let ABC be a Euclidean triangle with $\angle A = \angle B = \pi/4$ and $\angle C = \pi/2$. Let N be the midpoint of AB and let r_{CN} be a reflection with respect to the line CN. Show that $r_{CN} \circ R_{B,\pi/4} \circ R_{A,\pi/4}$ is a reflection.

Q2 (a) Prove or disprove that there exists an affine map $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $f(1,1) = (1,2), f(1,2) = (2,1), f(0,1) = (2,3)$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map. Assume that there exist $A = (a_1, a_2) \in \mathbb{R}^2$, $r > 0$, and an affine line $l \subset \mathbb{R}^2$ such that for

$$B_r(A) := \{x \in \mathbb{R}^2 | (x_1 - a_1)^2 + (x_2 - a_2)^2 < r^2\}$$

one has $f(B_r(A)) \subset l$. Prove that f is not affine.

Q3 Let $ABCD$ be a quadrilateral on the hyperbolic plane. Assume that $AB = BC = a$ and $\angle A = \angle B = \angle C = \pi/2$. Assume also that $D \in \partial \mathbb{H}^2$.

(a) Compute $\cosh a$.

(b) Find $\cosh d$, where d is the diameter of the circle inscribed into the triangle ACD.

SECTION B

Q4 (a) Prove or disprove that there exists a projective map $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ satisfying $f(1:1) = (1:-2),\ f(-2:5) = (-1:5),\ f(-1:-2) = (1:-4)$.

(b) Let $l := \{x_1 - x_2 = 0\} \subset \mathbb{RP}^2$ be a line in the projective plane and $A = (0:0:5),\ B = (1:1:1),\ C = (1:1:2),\ D = (1:1:3) \in l$ be four points on the line l. Compute the cross-ratio $[D, A, C, B]$.

(c) Let $m \subset \mathbb{RP}^2$ be a line and $P \in \mathbb{RP}^2$, $P \notin m$ be a point. Prove that there exists a projective map $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ with $f(m) = \{x_1 + x_2 + x_3 = 0\}$ and $f(P) = (2:1:2)$.

Q5 Let $D = (d_{ij})$, $i, j = 1, 2, 3$, be an invertible 3×3 matrix, and let the map $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ be defined by $f(x_1 : x_2 : x_3) = (\Sigma_j d_{1j} x_j : \Sigma_j d_{2j} x_j : \Sigma_j d_{3j} x_j)$.

(a) Prove that $(y_1 : y_2 : y_3) \in \mathbb{RP}^2$ is a fixed point of f if and only if (y_1, y_2, y_3) is an eigenvector of the matrix D.

(b) Let $l_1, l_2 \subset \mathbb{RP}^2$ be two lines, $l_1 \neq l_2$, and assume that $f(l_1) = l_1$ and $f(l_2) = l_2$. Is it true that l_1 always contains a fixed point? Justify your answer.

(c) The points $A_1, B_1, C_1, D_1 \in \mathbb{RP}^2$ are given in homogeneous coordinates by $A_1 = (1:3:1),\ B_1 = (3:1:1),\ C_1 = (2:2:1),\ D_1 = (1:1:0)$. Is there a projective transformation of $\mathbb{RP}^2$ which maps the points A_1, B_1, C_1, D_1 to $A_2 = (0:0:1),\ B_2 = (1:0:0),\ C_2 = (1:1:1),\ D_2 = (0:1:0)$ respectively? Justify your answer.

Q6 Three distinct circles C_1, C_2, C_3 on the Euclidean plane have a unique common point O. C_1 forms an angle $\pi/3$ with both other circles C_2 and C_3.

(a) Show that there exists either a circle or a line C tangent to each of C_1, C_2, C_3 simultaneously.

(b) Is it true or false that the circle or line C tangent to each of C_1, C_2, C_3 simultaneously is unique? Justify your answer.

(c) Let A_1 be the point of intersection of the circles C_2 and C_3 distinct from O. Define similarly A_2 and A_3 (as intersection points of circles C_1, C_3 and C_1, C_2 respectively).

Let γ be a circle or line passing through the points O and A_3, bisecting the angle between C_1 and C_2 and intersecting C_3 at a point B distinct from O. Find the cross-ratio $[O, A_2, B, A_1]$.

Q7 Let A and B be points on the hyperbolic plane and let $R_{A,\pi}$ and $R_{B,\pi}$ be rotations by π around these points. Let $\varphi = R_{B,\pi} \circ R_{A,\pi}$.

(a) Determine the type of the isometry φ. Justify your answer.

(b) Let r_{AB} be a reflection with respect to the line AB. Find the type of the isometry $r_{AB} \circ R_{A,\pi} \circ r_{AB}$. Justify your answer.

(c) Let $G = \langle \varphi, R_{A,\pi} \rangle$ be the group of isometries of $\mathbb{H}^2$ generated by φ and $R_{A,\pi}$. Is it true or false that G acts on $\mathbb{H}^2$ discretely (i.e. that the orbit of any point $P \in \mathbb{H}^2$ has no accumulation point in $\mathbb{H}^2$)? Justify your answer.

SECTION C

Q8 (a) An ellipse on the plane $\mathbb{R}^2$ with coordinates (x, y) is given by the equation $9x^2 + 4y^2 = 4$. Find the foci of this ellipse.

(b) Let two plane projective quadrics be given in homogeneous coordinates as $x_1^2 + 3x_1x_2 - x_2^2 = 0$ and $y_1^2 - y_2^2 - y_2y_3 = 0$. Prove or disprove that they are equivalent.