

EXAMINATION PAPER

Examination Session: May/June

Year: 2025

Exam Code:

MATH4241-WE01

Title:

Representation Theory IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions.				
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.				
	Write your answer in the white-covered answer booklet with barcodes.				
	Begin your answer to each question on a new page.				

Revision:

SECTION A

- **Q1** Let (π, V) be a finite dimensional complex representation of some finite group G and let $\langle \cdot, \cdot \rangle$ be a *G*-invariant inner product on *V*.
 - (a) Let (π, W) be a non-trivial subrepresentation of (π, V) and let

$$W^{\perp} = \{ \mathbf{v} \in V \, | \, \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W \},\$$

such that $V = W \oplus W^{\perp}$. Show that (π, W^{\perp}) is also a subrepresentation of (π, V) and thus that

$$(\pi, V) = (\pi, W) \oplus (\pi, W^{\perp}).$$

- (b) Using the definition of a character, show that $\chi_{(\pi,W^{\perp})} = \chi_{(\pi,V)} \chi_{(\pi,W)}$.
- **Q2** Let $\mathcal{M}^{(n-1,1)}$ be the complex vector space with basis

$$\left\{ \underbrace{\frac{1 \quad \dots \quad i-1 \quad i+1 \quad \dots \quad n}{\underline{i}}}_{i} \mid 1 \le i \le n \right\},$$

the set of Young tabloids of shape (n-1, 1), and let $(\pi, \mathcal{M}^{(n-1,1)})$ be the permutation representation of the symmetric group, S_n , arising from the natural action of S_n on this basis.

- (a) Show that $(\pi, \mathcal{M}^{(n-1,1)})$ is isomorphic to the standard permutation representation of S_n arising from the action of S_n on \mathbb{C}^n .
- (b) You may assume that

$$(\pi, \mathcal{M}^{(n-1,1)}) \cong (\pi, \mathcal{S}^{(n)}) \oplus (\pi, \mathcal{S}^{(n-1,1)}),$$

as S_n representations. Using this, or otherwise, compute $\operatorname{Res}_{S_{n-1}}^{S_n}(\pi, \mathcal{M}^{(n-1,1)})$. Give your answer as a sum of Specht modules.

- **Q3** (a) Write down the definition of U(1) and show that the Lie algebra of U(1) is $\mathfrak{u}_1 = \{i\theta \mid \theta \in \mathbb{R}\}.$
 - (b) What is the kernel of $\exp: \mathfrak{u}_1 \to \mathrm{U}(1)$?
 - (c) For which choice of $\lambda \in \mathbb{R}$ does the Lie algebra isomorphism

$$\phi:\mathfrak{u}_1\longrightarrow\mathfrak{u}_1\;;\;i\theta\longmapsto i\lambda\theta,$$

exponentiate to a Lie group homomorphism $\psi : U(1) \to U(1)$?

- **Q4** Let V be a, not necessarily finite dimensional, k-vector space, for k either \mathbb{R} or \mathbb{C} . Let $\operatorname{End}(V)$ be the k-vector space of linear maps from V to itself.
 - (a) Show that End(V) forms a Lie algebra with Lie bracket

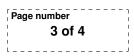
 $[\cdot, \cdot] : \operatorname{End}(V) \times \operatorname{End}(V) \longrightarrow \operatorname{End}(V),$

defined

$$[f,g] = f \circ g - g \circ f,$$

for all $f, g \in \text{End}(V)$. You may assume $[\cdot, \cdot]$ is bilinear.

(b) Let U be a subspace of V. Show that $\mathfrak{h} = \{f \in \operatorname{End}(V) \mid f(\mathbf{v}) \in U, \forall \mathbf{v} \in V\}$ is a Lie subalgebra of $\operatorname{End}(V)$.



SECTION B

Q5 Consider the group G with presentation

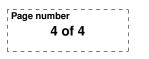
$$\langle r, s \mid r^8 = s^2 = 1, \, srs = r^3 \rangle,$$

often called the quasidihedral group. You may assume the order of G is 16.

- (a) Given an irreducible representation (π, V) of G you may assume we can pick a non-zero eigenvector \mathbf{v} of $\pi(r)$ with eigenvalue λ , i.e. $\pi(r)\mathbf{v} = \lambda \mathbf{v}$. Using this method, or otherwise, show that all irreducible representations of G have dimension at most 2.
- (b) Using part (a) find all 1-dimensional irreducible representations of G.
- (c) Find all the remaining irreducible representations of G.
- **Q6** Let G be a group of order 24. The conjugacy classes of G and their sizes are shown below. The classes are labelled according to the order of their elements, for example C_{3A} and C_{3B} are two different classes consisting of elements of order 3. You are also given two of the irreducible characters of G and some values of another irreducible character. Here $\omega = \exp(2\pi i/3)$.

class	\mathcal{C}_1	\mathcal{C}_2	\mathcal{C}_{3A}	\mathcal{C}_{3B}	\mathcal{C}_4	\mathcal{C}_{6A}	\mathcal{C}_{6B}
class size	1	1	4	4	6	4	4
χ_{π}	1	1	ω	ω^2 -1	1	ω	ω^2
$\chi_{ ho}$	2	-2	-1	-1	0	1	1
χ_{lpha}			$-\omega$	$-\omega^2$		ω	ω^2

- (a) You may assume the union of conjugacy classes $N = C_1 \cup C_2 \cup C_4$ is a normal subgroup in G and $G/N \cong C_3$, the cyclic group of order 3. By finding the character table of C_3 and then lifting characters, or otherwise, find two other 1-dimensional characters of G.
- (b) How many irreducible representations of G are there and what are the remaining unknown dimensions? Give justification for your answer.
- (c) Compute $\chi_{\text{Sym}^2\rho}$ and use it to complete the character table of G.





- **Q7** Let G be a linear Lie group and let \mathfrak{g} be its Lie algebra. The Killing form is a bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by $\kappa(A, B) = \operatorname{tr}(\operatorname{ad}_A \circ \operatorname{ad}_B)$, for all $A, B \in \mathfrak{g}$, where $\operatorname{ad}_A \in \operatorname{End}(\mathfrak{g})$ is the linear map corresponding to the adjoint representation of \mathfrak{g} at $A \in \mathfrak{g}$.
 - (a) Consider $\mathfrak{sl}_{2,\mathbb{C}}$ with the standard basis $\{X, Y, H\}$.
 - (i) Compute the matrices of ad_X , ad_Y and ad_H with respect to the standard basis.
 - (ii) Show that κ is non-zero on $\mathfrak{sl}_{2,\mathbb{C}}$.
 - (b) Let $\phi_g : \mathfrak{g} \to \mathfrak{g}$ be the invertible linear map given by $Z \mapsto gZg^{-1}$, for $g \in G$. Show that

$$\operatorname{ad}_{gAg^{-1}} \circ \operatorname{ad}_{gBg^{-1}} = \phi_g \circ \operatorname{ad}_A \circ \operatorname{ad}_B \circ \phi_q^{-1},$$

for all $A, B \in \mathfrak{g}$, as elements in $\operatorname{End}(\mathfrak{g})$.

(c) Hence, or otherwise, show that κ is invariant under the adjoint action of G, i.e. for all $g \in G$ and all $A, B \in \mathfrak{g}$

$$\kappa(\operatorname{Ad}_g(A), \operatorname{Ad}_g(B)) = \kappa(A, B).$$

- **Q8** Let $\operatorname{Sym}^{l}(\mathbb{C}^{2})$ be the *l*-th symmetric product of the standard representation of $\mathfrak{sl}_{2,\mathbb{C}}$, for $l \in \mathbb{N}$. Consider $\bigwedge^{2} \operatorname{Sym}^{l}(\mathbb{C}^{2})$.
 - (a) Show that all the weights of $\bigwedge^2 \operatorname{Sym}^l(\mathbb{C}^2)$ are even.
 - (b) By considering the weight 0 vectors of $\bigwedge^2 \operatorname{Sym}^l(\mathbb{C}^2)$ deduce that the number of irreducible constituents of $\bigwedge^2 \operatorname{Sym}^l(\mathbb{C}^2)$ is l/2 when l is even and (l+1)/2 when l is odd.
 - (c) Let l = 2r be even. Show, by induction on r or otherwise, that

$$\bigwedge^2 \operatorname{Sym}^l(\mathbb{C}^2) \cong \bigoplus_{i=1}^{l/2} \operatorname{Sym}^{4i-2}\mathbb{C}^2.$$