Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH4241-WE01

Title: Representation Theory IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Let (π, V) be a finite dimensional complex representation of some finite group G and let $\langle \cdot, \cdot \rangle$ be a G-invariant inner product on V.

(a) Let (π, W) be a non-trivial subrepresentation of (π, V) and let

$$W^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\},$$

such that $V = W \oplus W^{\perp}$. Show that $(\pi, W^{\perp})$ is also a subrepresentation of (π, V) and thus that

$$(\pi, V) = (\pi, W) \oplus (\pi, W^{\perp}).$$

(b) Using the definition of a character, show that $\chi_{(\pi,W^{\perp})} = \chi_{(\pi,V)} - \chi_{(\pi,W)}$.

Q2 Let $\mathcal{M}^{(n-1,1)}$ be the complex vector space with basis

$$\left\{ \begin{array}{l} \overline{\underline{1 \ \ \ldots \ \ i{-}1 \ \ i{+}1 \ \ \ldots \ \ n}} \\ \underline{i} \end{array} \;\middle|\; 1 \leq i \leq n \right\},$$

the set of Young tabloids of shape $(n-1, 1)$, and let $(\pi, \mathcal{M}^{(n-1,1)})$ be the permutation representation of the symmetric group, S_n, arising from the natural action of S_n on this basis.

(a) Show that $(\pi, \mathcal{M}^{(n-1,1)})$ is isomorphic to the standard permutation representation of S_n arising from the action of S_n on $\mathbb{C}^n$.

(b) You may assume that

$$(\pi, \mathcal{M}^{(n-1,1)}) \cong (\pi, \mathcal{S}^{(n)}) \oplus (\pi, \mathcal{S}^{(n-1,1)}),$$

as S_n representations. Using this, or otherwise, compute $\mathrm{Res}^{S_n}_{S_{n-1}}(\pi, \mathcal{M}^{(n-1,1)})$. Give your answer as a sum of Specht modules.

Q3 (a) Write down the definition of $\mathrm{U}(1)$ and show that the Lie algebra of $\mathrm{U}(1)$ is $\mathfrak{u}_1 = \{i\theta \mid \theta \in \mathbb{R}\}$.

(b) What is the kernel of $\exp : \mathfrak{u}_1 \to \mathrm{U}(1)$?

(c) For which choice of $\lambda \in \mathbb{R}$ does the Lie algebra isomorphism

$$\phi : \mathfrak{u}_1 \longrightarrow \mathfrak{u}_1 \ ; \ i\theta \longmapsto i\lambda\theta,$$

exponentiate to a Lie group homomorphism $\psi : \mathrm{U}(1) \to \mathrm{U}(1)$?

Q4 Let V be a, not necessarily finite dimensional, k-vector space, for k either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathrm{End}(V)$ be the k-vector space of linear maps from V to itself.

(a) Show that $\mathrm{End}(V)$ forms a Lie algebra with Lie bracket

$$[\cdot, \cdot] : \mathrm{End}(V) \times \mathrm{End}(V) \longrightarrow \mathrm{End}(V),$$

defined

$$[f, g] = f \circ g - g \circ f,$$

for all $f, g \in \mathrm{End}(V)$. You may assume $[\cdot, \cdot]$ is bilinear.

(b) Let U be a subspace of V. Show that $\mathfrak{h} = \{f \in \mathrm{End}(V) \mid f(\mathbf{v}) \in U, \forall \mathbf{v} \in V\}$ is a Lie subalgebra of $\mathrm{End}(V)$.

SECTION B

Q5 Consider the group G with presentation

$$\langle r, s \,|\, r^8 = s^2 = 1,\ srs = r^3 \rangle,$$

often called the quasidihedral group. You may assume the order of G is 16.

(a) Given an irreducible representation (π, V) of G you may assume we can pick a non-zero eigenvector $\mathbf{v}$ of $\pi(r)$ with eigenvalue λ, i.e. $\pi(r)\mathbf{v} = \lambda\mathbf{v}$. Using this method, or otherwise, show that all irreducible representations of G have dimension at most 2.

(b) Using part (a) find all 1-dimensional irreducible representations of G.

(c) Find all the remaining irreducible representations of G.

Q6 Let G be a group of order 24. The conjugacy classes of G and their sizes are shown below. The classes are labelled according to the order of their elements, for example $\mathcal{C}_{3A}$ and $\mathcal{C}_{3B}$ are two different classes consisting of elements of order 3. You are also given two of the irreducible characters of G and some values of another irreducible character. Here $\omega = \exp(2\pi i/3)$.

class	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_{3A}$	$\mathcal{C}_{3B}$	$\mathcal{C}_4$	$\mathcal{C}_{6A}$	$\mathcal{C}_{6B}$
size	1	1	4	4	6	4	4
χ_π	1	1	ω	ω^2	1	ω	ω^2
χ_ρ	2	-2	-1	-1	0	1	1
χ_α			$-\omega$	$-\omega^2$		ω	ω^2

(a) You may assume the union of conjugacy classes $N = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_4$ is a normal subgroup in G and $G/N \cong C_3$, the cyclic group of order 3. By finding the character table of C_3 and then lifting characters, or otherwise, find two other 1-dimensional characters of G.

(b) How many irreducible representations of G are there and what are the remaining unknown dimensions? Give justification for your answer.

(c) Compute $\chi_{\mathrm{Sym}^2\rho}$ and use it to complete the character table of G.

Q7 Let G be a linear Lie group and let $\mathfrak{g}$ be its Lie algebra. The *Killing form* is a bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by $\kappa(A, B) = \mathrm{tr}(\mathrm{ad}_A \circ \mathrm{ad}_B)$, for all $A, B \in \mathfrak{g}$, where $\mathrm{ad}_A \in \mathrm{End}(\mathfrak{g})$ is the linear map corresponding to the adjoint representation of $\mathfrak{g}$ at $A \in \mathfrak{g}$.

(a) Consider $\mathfrak{sl}_{2,\mathbb{C}}$ with the standard basis $\{X, Y, H\}$.

(i) Compute the matrices of ad_X, ad_Y and ad_H with respect to the standard basis.

(ii) Show that κ is non-zero on $\mathfrak{sl}_{2,\mathbb{C}}$.

(b) Let $\phi_g : \mathfrak{g} \to \mathfrak{g}$ be the invertible linear map given by $Z \mapsto gZg^{-1}$, for $g \in G$. Show that

$$\mathrm{ad}_{gAg^{-1}} \circ \mathrm{ad}_{gBg^{-1}} = \phi_g \circ \mathrm{ad}_A \circ \mathrm{ad}_B \circ \phi_g^{-1},$$

for all $A, B \in \mathfrak{g}$, as elements in $\mathrm{End}(\mathfrak{g})$.

(c) Hence, or otherwise, show that κ is invariant under the adjoint action of G, i.e. for all $g \in G$ and all $A, B \in \mathfrak{g}$

$$\kappa(\mathrm{Ad}_g(A), \mathrm{Ad}_g(B)) = \kappa(A, B).$$

Q8 Let $\mathrm{Sym}^l(\mathbb{C}^2)$ be the l-th symmetric product of the standard representation of $\mathfrak{sl}_{2,\mathbb{C}}$, for $l \in \mathbb{N}$. Consider $\bigwedge^2 \mathrm{Sym}^l(\mathbb{C}^2)$.

(a) Show that all the weights of $\bigwedge^2 \mathrm{Sym}^l(\mathbb{C}^2)$ are even.

(b) By considering the weight 0 vectors of $\bigwedge^2 \mathrm{Sym}^l(\mathbb{C}^2)$ deduce that the number of irreducible constituents of $\bigwedge^2 \mathrm{Sym}^l(\mathbb{C}^2)$ is $l/2$ when l is even and $(l+1)/2$ when l is odd.

(c) Let $l = 2r$ be even. Show, by induction on r or otherwise, that

$$\bigwedge^2 \mathrm{Sym}^l(\mathbb{C}^2) \cong \bigoplus_{i=1}^{l/2} \mathrm{Sym}^{4i-2}\mathbb{C}^2.$$