Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH4261-WE01

Title:
Stochastic Analysis IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 Let $(W_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$.

(a) Justify your answers as to whether each of the following stochastic processes is a Brownian motion or not.

(i) $X_t := 2W_{t/4}$

(ii) $Y_t := W_{2t} - W_t$

(iii) $Z_t := \sqrt{t}W_1$

(b) Prove that the following stochastic process is a Brownian motion:

$$B_t := \begin{cases} tW_{1/t}, & t > 0 \\ 0, & t = 0 \end{cases}$$

Hint: Carefully discuss the continuity of $t \mapsto B_t$ at $t = 0$.

Q2 (a) Let $X = (X_n)_{n\in\mathbb{Z}_+}$ be a discrete-time martingale with respect to a filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, such that $X_0 = 0$. Show that

$$\mathbb{E}[X_n^2] = \sum_{k=0}^{n-1} \mathbb{E}\big[(X_{k+1} - X_k)^2\big]$$

for each $n \in \mathbb{N}$.

Now let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables, such that

$$\mathbb{E}[Z_n] = 0 \qquad \text{and} \qquad \mathrm{Var}(Z_n) = \gamma^n$$

for each $n \in \mathbb{N}$, for some $\gamma \in (0, 1)$, and let $\mathcal{F}_n = \sigma(Z_1, Z_2, \dots, Z_n)$, $n \in \mathbb{Z}_+$, be the natural filtration of this sequence. Let $M_0 = 0$, and

$$M_n = \sum_{k=1}^{n} Z_k$$

for each $n \in \mathbb{N}$.

(b) Show that $M = (M_n)_{n\in\mathbb{Z}_+}$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$.

(c) Use the result of part (a) to show that M is bounded in L^2.

Q3 In your answers to the following parts, you may use, without proof, the result that any bounded and continuous real-valued martingale M is of finite quadratic variation and $\langle M, M\rangle$ is the unique continuous, adapted and increasing real-valued process such that $M^2 - \langle M, M\rangle$ is a martingale.

(a) Let $M_t, t \in \mathbb{R}_+$, be a continuous real-valued local martingale. Prove that there is a unique continuous increasing adapted process $\langle M, M\rangle$, vanishing at zero, such that $M^2 - \langle M, M\rangle$ is a continuous local martingale.

(b) Show that if $M_t, t \in \mathbb{R}_+$, is a continuous and bounded real-valued martingale, then

$$\mathbb{E}\big[(M_t - M_0)^2\big] = \mathbb{E}\big[\langle M, M\rangle_t\big].$$

Q4 (a) Let B be a Brownian motion on $\mathbb{R}$, and K be a progressive process on the same filtered probability space satisfying, for every $t > 0$,

$$\mathbb{E}\left[\int_0^t K_s^2\, ds\right] < \infty.$$

Denote by H^2 the set of the L^2-bounded and continuous martingales (i.e., all continuous martingales M satisfying $\sup_{t\in\mathbb{R}_+} \mathbb{E}[M_t^2] < \infty$), and H_0^2 the subset of H^2 vanishing at 0.

Use the stopping technique and stochastic integration theory on H_0^2 for a progressive process, say $\widetilde{K}$, satisfying

$$\mathbb{E}\left[\int_0^\infty \widetilde{K}_s^2\, ds\right] < \infty,$$

to construct the stochastic integral $\int_0^t K_s\, dB_s$ for all $t > 0$.

(b) Verify that you can use part (a) of this question to define $M_t = \int_0^t B_s\, dB_s$.

(c) Calculate $\langle M, M\rangle_t$ and $\mathbb{E}[M_t^2]$.

SECTION B

Q5 (a) State the definition that a continuous function $u\colon [0,T] \to \mathbb{R}$ has a finite quadratic variation.

(b) Let $u\colon [0,T] \to \mathbb{R}$ be a C^1-function. Prove that its quadratic variation is zero.

(c) Let $(W_t)_{t\in[0,T]}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define

$$[W]_{\mathbf{t},T} := \sum_{i=0}^{n_{\mathbf{t}}-1} |W_{t_{i+1}} - W_{t_i}|^2,$$

where $\mathbf{t} = (t_i)_{i=0}^{n_{\mathbf{t}}}$ with $0 = t_0 < t_1 < \cdots < t_{n_{\mathbf{t}}} = T$ is any partition of $[0,T]$. Prove that

$$\lim_{|\mathbf{t}|\to 0} [W]_{\mathbf{t},T} = T \quad \text{in} \quad L^2(\Omega, \mathbb{P}).$$

Q6 Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables, such that

$$\mathbb{P}(\xi_i = 1) = p \qquad \text{and} \qquad \mathbb{P}(\xi_i = -1) = 1 - p$$

for some $p \in (0, 1)$, and let $\mathcal{F}_n = \sigma(\xi_1, \xi_2, \ldots, \xi_n)$, $n \in \mathbb{Z}_+$, be the natural filtration of this sequence. Let $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} \xi_i$$

for each $n \in \mathbb{N}$. Let

$$T = \inf\{n \in \mathbb{Z}_+ : S_n = k\}$$

for some $k \in \mathbb{N}$, and let $X_n = 2^{S_n}$ for each $n \in \mathbb{Z}_+$.

(a) Show that the process $X = (X_n)_{n\in\mathbb{Z}_+}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ for a particular value of p, which you should identify.

(b) State Doob's supermartingale convergence theorem, and then use it to show, with the value of p found in part (a), that the limit

$$\lim_{n\to\infty} X_{n\wedge T}$$

exists almost surely.

(c) Show that this limit is given by

$$\lim_{n\to\infty} X_{n\wedge T} = 2^k \mathbb{1}_{\{T<\infty\}}.$$

(d) Using the result of part (c), prove that

$$\mathbb{P}\Big(\sup_{n\in\mathbb{Z}_+} S_n \geq k\Big) = 2^{-k}.$$

Q7 Let $X_t, t \geq 0$, be a real-valued submartingale such that $\mathbb{E}[|X_t|] < \infty$ for every $t \geq 0$. For some $t \in (0, \infty)$, let $t_n, n = 1, 2, \ldots$, be a sequence decreasing to t, i.e., such that $t_n \searrow t$ as $n \to \infty$.

(a) Use the submartingale downcrossing inequality to prove that

$$X_{t+}(\omega) := \lim_{n\to\infty} X_{t_n}(\omega)$$

exists for almost every $\omega \in \Omega$.

(b) Prove that $\sup_{n\in\mathbb{N}} \mathbb{E}[|X_{t_n}|] < \infty$.

(c) Show that $X_{t_n}, n = 1, 2, \ldots$, is uniformly integrable with respect to expectation.

(d) Prove that $\mathbb{E}[|X_{t+}|] < \infty$.

Q8 Let X^x_t, $t \in \mathbb{R}_+$, be the solution of the following stochastic differential equation

$$dX^x_t = -X^x_t \, dt + dW_t, \qquad X^x_0 = x,$$

where W_t, $t \in \mathbb{R}_+$, is a standard Brownian motion on $\mathbb{R}$, and $x \in \mathbb{R}$. For any bounded measurable function $f\colon \mathbb{R} \to \mathbb{R}$, let

$$P_t f(x) = \mathbb{E}\big[f(X^x_t)\big], \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}.$$

(a) Prove, using the Markov property, that for any $t, s \geq 0$, and f being bounded and measurable,

$$P_t \circ P_s f = P_{t+s} f.$$

(b) Verify that X^x_t, $t > 0$, is given by

$$X^x_t = \exp\{-t\}x + \int_0^t \exp\{-(t-s)\} \, dW_s.$$

And use this to show that for any $t > 0$, X^x_t is a Gaussian random variable with mean $e^{-t}x$ and variance $\frac{1}{2}(1 - e^{-2t})$, and

$$P_t f(x) = \frac{1}{\sqrt{\pi(1-e^{-2t})}} \int_{\mathbb{R}} f(y) \exp\Big\{ -\frac{(y - e^{-t}x)^2}{1 - e^{-2t}} \Big\} \, dy.$$

(c) Let A be the infinitesimal generator of X^x_t, $t \in \mathbb{R}_+$. Prove that

$$Af = \frac{1}{2} \frac{d^2}{dx^2} f - x \frac{d}{dx} f$$

when $f \in C^2$ is bounded and has bounded continuous derivatives up to second order.