Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH4271-WE01

Title: Superstrings IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 Consider the following action for a p-dimensional membrane:

$$S = \int d^{p+1}\sigma\sqrt{-h}\left(-\frac{1}{2}T_p h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X + \Lambda_p\right), \quad (1)$$

where σ^α with $\alpha \in \{0, 1, ..., p\}$, are worldvolume coordinates, $X^\mu(\sigma^0, ..., \sigma^p)$ describe the embedding into the target space, T_p is the tension of the membrane, $h_{\alpha\beta}$ is the worldvolume metric, Λ_p is a worldvolume cosmological constant, and

$$\partial_\alpha X \cdot \partial_\beta X = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \quad (2)$$

(a) Show that the equation of motion for $h_{\alpha\beta}$ is

$$T_p\left(\partial_\gamma X \cdot \partial_\delta X - \frac{1}{2}h_{\gamma\delta}h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X\right) + \Lambda_p h_{\gamma\delta} = 0. \quad (3)$$

Hint: Recall that $\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}h_{\alpha\beta}\delta h^{\alpha\beta}$.

(b) Let us make the following ansatz:

$$h_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X. \quad (4)$$

Show that this is a solution to (3) if

$$\Lambda_p = \frac{1}{2}(p-1)T_p. \quad (5)$$

(c) Using (4) and (5), show that the action in (1) is classically equivalent to the Nambu-Goto action for a p-dimensional membrane:

$$S = -T_p\int d^{p+1}\sigma\sqrt{-\det G_{\alpha\beta}}, \;\; G_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X. \quad (6)$$

Q2 (a) The tree-level amplitude of four tachyons in closed bosonic string theory is given by

$$\mathcal{A}_4^{\mathrm{cl}}(s,t,u) \propto B\left(-\frac{\alpha'}{4}s-1,-\frac{\alpha'}{4}t-1,-\frac{\alpha'}{4}u-1\right), \tag{7}$$

where we neglect a numerical prefactor,

$$B(a,b,c) = \pi\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(b+c)\Gamma(c+a)}, \tag{8}$$

and s,t,u are the Mandelstam variables for all outgoing momenta:

$$s=-(p_1+p_2)^2,\ \ t=-(p_1+p_4)^2,\ \ s=-(p_1+p_3)^2. \tag{9}$$

For what values of s does the amplitude have a pole? What is the interpretation of these poles?

(b) The tree-level amplitude of four tachyons in open bosonic string theory with Neumann boundary conditions is given by

$$\mathcal{A}_4^{\mathrm{op}}(s,t) \propto B\left(-\alpha's-1,-\alpha't-1\right), \tag{10}$$

where we have neglected a numerical prefactor and

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{11}$$

For what values of s does the amplitude have a pole? What is the interpretation of these poles?

Q3 Consider a Dirac spinor representation with Lorentz generators $S^{\mu\nu} = \frac{1}{4}[\Gamma^\mu, \Gamma^\nu]$ and assume there is constant matrix B such that:

$$(S^{\mu\nu})^* = B S^{\mu\nu} B^{-1} \,,$$

where A^* denotes the complex conjugate of the matrix A. Show that for a spinor χ_M, the Majorana condition $\chi_M^* = B\chi_M$ is compatible with the action of the Lorentz group.

Q4 Consider a system with 2 fermionic creation/annihilation operators, $\hat{b}_i^\dagger, \hat{b}_i$, with canonical anti-commutation relations:

$$\{\hat{b}_i, \hat{b}_j^\dagger\} = \delta_{ij} \,, \qquad \{\hat{b}_i, \hat{b}_j\} = \{\hat{b}_i^\dagger, \hat{b}_j^\dagger\} = 0 \,,$$

for $i, j \in \{1, 2\}$. Define the operator

$$\hat{O} = \exp\left(\alpha\, \hat{b}_1^\dagger \hat{b}_1 + \beta\, \hat{b}_2^\dagger \hat{b}_2\right) \,,$$

with $\alpha, \beta \in \mathbb{C}$ two complex numbers and show that

$$\mathcal{O} = 1 + \alpha\, \hat{b}_1^\dagger \hat{b}_1 + \beta\, \hat{b}_2^\dagger \hat{b}_2 - 2\alpha\beta \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_1 \hat{b}_2 \,.$$

SECTION B

Q5 A general solution to the equations of motion for the bosonic string is given by $X^\mu(\tau,\sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$, where $\sigma^\pm = \tau \pm \sigma$ and

$$X_L^\mu\left(\sigma^+\right) = a^\mu + b^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+} \tag{12}$$

$$X_R^\mu\left(\sigma^-\right) = c^\mu + d^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-} \tag{13}$$

In the following, you will be asked to deduce constraints on $a^\mu, b^\mu, c^\mu, d^\mu, \alpha_n^\mu, \tilde{\alpha}_n^\mu$ for various boundary conditions.

(a) Consider an open string which has Neumann boundary conditions for X^i where $i \in \{0, ..., 24\}$ and momentum p^i along those directions. Find the mode expansion for X^i. Recall that $0 \leq \sigma \leq \pi$ and the momentum is given by

$$p^\mu = T\int_0^\pi d\sigma\partial_\tau X^\mu, \;\; T = 1/2\pi\alpha'. \tag{14}$$

(b) Suppose the string has the following boundary conditions for X^{25}:

$$X^{25}(\tau,\sigma=0) = 0, \;\; X^{25}(\tau,\sigma=\pi) = L. \tag{15}$$

Find the mode expansion for X^{25}. What is the momentum along the X^{25} direction?

(c) Find the classical mass of the string if one sets all of the oscillator modes to zero. Hint: use the Virasoro constraints.

Q6 The Euclidean worldsheet for an open string can be mapped to the upper half of the complex plane using the following coordinates:

$$z = e^{\tau - i\sigma}, \;\; \bar{z} = e^{\tau + i\sigma}, \;\; 0 \leq \sigma \leq \pi. \tag{16}$$

(a) Show that the boundaries of the worldsheet correspond to $z = \bar{z}$.

(b) Show that Neumann boundary conditions correspond to the constraint

$$\left(z\partial - \bar{z}\bar{\partial}\right) X^{\mu}(z, \bar{z})\Big|_{z=\bar{z}} = 0, \tag{17}$$

where $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$.

(c) For open strings with Neumann boundary conditions, the worldsheet fields have the following operator product expansion:

$$X(z, \bar{z})X(w, \bar{w}) = G(z, \bar{z}; w, \bar{w}) + ..., \tag{18}$$

where we take the target space to be one-dimensional for simplicity,

$$G(z, \bar{z}; w, \bar{w}) = -\frac{\alpha'}{2} \ln |z - w|^2 - \frac{\alpha'}{2} \ln |z - \bar{w}|^2 , \tag{19}$$

and "..." denote non-singular terms. Show that $G(z, \bar{z}; w, \bar{w})$ satisfies (17).

(d) Using (18), compute the singular part of the operator product expansion of $\partial X(z)$ with $: e^{ipX(w,\bar{w})} :$.

(e) Consider the vertex operator

$$V = \int_{\partial \mathcal{M}} ds \hat{V}(s), \tag{20}$$

where the integral is over the boundary of the worldsheet, which we parameterise by the variable s, and $\hat{V}(s) = \, : e^{ipX(w,\bar{w})} :\big|_{w=\bar{w}=s}$. What must the mass of this state be in order for the vertex operator to be well-defined? What is the physical interpretation of this state? Hint: deduce the conformal weight of this state from the operator product expansion of $T(z) = -\frac{1}{\alpha'} : \partial X(z)\partial X(z) :$ with $: e^{ipX(w,\bar{w})} :$ restricted to the boundary.

Q7 The Hamiltonian for a quantum mechanical system takes the form

$$\hat{H} = \frac{1}{2}\{\hat{Q}, \hat{Q}^\dagger\}\,,$$

with $\hat{Q}$ a fermionic operator such that $\hat{Q}^2 = (\hat{Q}^\dagger)^2 = 0$.

(a) Check that the operators $\hat{Q}$ and $\hat{Q}^\dagger$ are symmetries of the system, i.e. they commute with the Hamiltonian.

(b) Show that the only possible eigenvalue for $\hat{Q}$ and $\hat{Q}^\dagger$ is zero.

(c) Consider a state $|\psi\rangle$ such that $\hat{Q}\,|\psi\rangle = 0$ and $\hat{H}\,|\psi\rangle = E\,|\psi\rangle$ with $E \neq 0$. Find a second state $|\chi\rangle$ such that $\hat{H}\,|\chi\rangle = E\,|\chi\rangle$. Why does this argument fail when $E = 0$?

Q8 Consider the fermionic zero-modes in the Ramond sector of the RNS superstring given by the operators $\hat{d}_0^i$ with $i \in \{2, ..., 9\}$ which obey the algebra

$$\{\hat{d}_0^i, \hat{d}_0^j\} = \delta^{ij} .$$

Upon rescaling $\gamma^i \doteq \sqrt{2}\hat{d}_0^i$ we obtain the Clifford algebra in 8 Euclidean dimensions:

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} ,$$

and then define

$$A^a_\pm \doteq \frac{1}{2}(\gamma^{2a} \pm i\gamma^{2a+1}) ,$$

with $a \in \{1, .., 4\}$.

(a) Show that the operators $A^a_\pm$ satisfy

$$\{A^a_-, A^b_+\} = \delta^{ab} \qquad ; \qquad \{A^a_+, A^b_+\} = \{A^a_-, A^b_-\} = 0 .$$

(b) Define the chirality operator

$$\Gamma = \prod_{i=2}^{9} \gamma^i = \gamma^2 \cdots \gamma^9 ,$$

and show that $\{\Gamma, A^a_+\} = 0$ for all $a \in \{1, ..., 4\}$.

(c) Define the vacuum state $|0\rangle_{\mathsf{R}}$ by imposing $A^a_- |0\rangle_{\mathsf{R}} = 0$ for all $a \in \{1, ..., 4\}$. Show that $\Gamma |0\rangle_{\mathsf{R}} = + |0\rangle_{\mathsf{R}}$ and that on the vector subspace

$$V_{8_c} = \text{span}\{A^{a_1}_+ |0\rangle_{\mathsf{R}} ,\ A^{a_1}_+ A^{a_2}_+ A^{a_3}_+ |0\rangle_{\mathsf{R}}\}$$

the operator Γ acts as $-\hat{I}$ with $\hat{I}$ the identity operator.