

EXAMINATION PAPER

Examination Session: May/June

2025

Year:

Exam Code:

MATH4271-WE01

Title:

Superstrings IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions.		
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.		
	Write your answer in the white-covered answer booklet with barcodes.		
	Begin your answer to each question on a new page.		

Revision:

SECTION A

Q1 Consider the following action for a p-dimensional membrane:

$$S = \int d^{p+1}\sigma \sqrt{-h} \left(-\frac{1}{2}T_p h^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X + \Lambda_p \right), \tag{1}$$

where σ^{α} with $\alpha \in \{0, 1, ..., p\}$, are worldvolume coordinates, $X^{\mu}(\sigma^{0}, ..., \sigma^{p})$ describe the embedding into the target space, T_{p} is the tension of the membrane, $h_{\alpha\beta}$ is the worldvolume metric, Λ_{p} is a worldvolume cosmological constant, and

 $\partial_{\alpha} X \cdot \partial_{\beta} X = \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} \eta_{\mu\nu}. \tag{2}$

(a) Show that the equation of motion for $h_{\alpha\beta}$ is

$$T_p\left(\partial_{\gamma}X\cdot\partial_{\delta}X - \frac{1}{2}h_{\gamma\delta}h^{\alpha\beta}\partial_{\alpha}X\cdot\partial_{\beta}X\right) + \Lambda_p h_{\gamma\delta} = 0.$$
(3)

Hint: Recall that $\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}h_{\alpha\beta}\delta h^{\alpha\beta}$.

(b) Let us make the following ansatz:

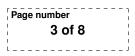
$$h_{\alpha\beta} = \partial_{\alpha} X \cdot \partial_{\beta} X. \tag{4}$$

Show that this is a solution to (3) if

$$\Lambda_p = \frac{1}{2}(p-1)T_p. \tag{5}$$

(c) Using (4) and (5), show that the action in (1) is classically equivalent to the Nambu-Goto action for a p-dimensional membrane:

$$S = -T_p \int d^{p+1}\sigma \sqrt{-\det G_{\alpha\beta}}, \ G_{\alpha\beta} = \partial_{\alpha} X \cdot \partial_{\beta} X.$$
 (6)





 ${\bf Q2}$ (a) The tree-level amplitude of four tachyons in closed bosonic string theory is given by

$$\mathcal{A}_4^{\rm cl}(s,t,u) \propto B\left(-\frac{\alpha'}{4}s-1, -\frac{\alpha'}{4}t-1, -\frac{\alpha'}{4}u-1\right),\tag{7}$$

where we neglect a numerical prefactor,

$$B(a,b,c) = \pi \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(b+c)\Gamma(c+a)},$$
(8)

and s, t, u are the Mandelstam variables for all outgoing momenta:

$$s = -(p_1 + p_2)^2, \ t = -(p_1 + p_4)^2, \ s = -(p_1 + p_3)^2.$$
 (9)

For what values of s does the amplitude have a pole? What is the interpretation of these poles?

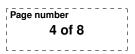
(b) The tree-level amplitude of four tachyons in open bosonic string theory with Neumann boundary conditions is given by

$$\mathcal{A}_4^{\rm op}(s,t) \propto B\left(-\alpha' s - 1, -\alpha' t - 1\right),\tag{10}$$

where we have neglected a numerical prefactor and

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$
(11)

For what values of s does the amplitude have a pole? What is the interpretation of these poles?



Q3 Consider a Dirac spinor representation with Lorentz generators $S^{\mu\nu} = \frac{1}{4}[\Gamma^{\mu}, \Gamma^{\nu}]$ and assume there is constant matrix *B* such that:

$$(S^{\mu\nu})^* = BS^{\mu\nu}B^{-1},$$

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where A^* denotes the complex conjugate of the matrix A. Show that for a spinor χ_M , the Majorana condition $\chi_M^* = B\chi_M$ is compatible with the action of the Lorentz group.

Q4 Consider a system with 2 fermionic creation/annihilation operators, $\hat{b}_i^{\dagger}, \hat{b}_i$, with canonical anti-commutation relations:

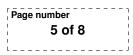
$$\{\hat{b}_i, \hat{b}_j^{\dagger}\} = \delta_{ij}, \qquad \{\hat{b}_i, \hat{b}_j\} = \{\hat{b}_i^{\dagger}, \hat{b}_j^{\dagger}\} = 0,$$

for $i, j \in \{1, 2\}$. Define the operator

$$\hat{O} = \exp\left(\alpha\,\hat{b}_1^{\dagger}\hat{b}_1 + \beta\,\hat{b}_2^{\dagger}\hat{b}_2\right)\,,$$

with $\alpha, \beta \in \mathbb{C}$ two complex numbers and show that

$$\mathcal{O} = 1 + \alpha \, \hat{b}_1^{\dagger} \hat{b}_1 + \beta \, \hat{b}_2^{\dagger} \hat{b}_2 - 2\alpha \beta \hat{b}_1^{\dagger} \hat{b}_2^{\dagger} \hat{b}_1 \hat{b}_2 \,.$$





SECTION B

Q5 A general solution to the equations of motion for the bosonic string is given by $X^{\mu}(\tau, \sigma) = X^{\mu}_{L}(\sigma^{+}) + X^{\mu}_{R}(\sigma^{-})$, where $\sigma^{\pm} = \tau \pm \sigma$ and

$$X_L^{\mu}\left(\sigma^{+}\right) = a^{\mu} + b^{\mu}\sigma^{+} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{\mu}e^{-in\sigma^{+}}$$
(12)

$$X_{R}^{\mu}(\sigma^{-}) = c^{\mu} + d^{\mu}\sigma^{+} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_{n}^{\mu}e^{-in\sigma^{-}}$$
(13)

In the following, you will be asked to deduce constraints on $a^{\mu}, b^{\mu}, c^{\mu}, d^{\mu}, \alpha^{\mu}_{n}, \tilde{\alpha}^{\mu}_{n}$ for various boundary conditions.

(a) Consider an open string which has Neumann boundary conditions for X^i where $i \in \{0, ..., 24\}$ and momentum p^i along those directions. Find the mode expansion for X^i . Recall that $0 \le \sigma \le \pi$ and the momentum is given by

$$p^{\mu} = T \int_0^{\pi} d\sigma \partial_{\tau} X^{\mu}, \quad T = 1/2\pi \alpha'.$$
(14)

(b) Suppose the string has the following boundary conditions for X^{25} :

$$X^{25}(\tau, \sigma = 0) = 0, \ X^{25}(\tau, \sigma = \pi) = L.$$
(15)

Find the mode expansion for X^{25} . What is the momentum along the X^{25} direction?

(c) Find the classical mass of the string if one sets all of the oscillator modes to zero. Hint: use the Virasoro constraints.



Q6 The Euclidean worldsheet for an open string can be mapped to the upper half of the complex plane using the following coordinates:

$$z = e^{\tau - i\sigma}, \quad \bar{z} = e^{\tau + i\sigma}, \quad 0 \le \sigma \le \pi.$$
(16)

- (a) Show that the boundaries of the worldsheet correspond to $z = \overline{z}$.
- (b) Show that Neumann boundary conditions correspond to the constraint

$$\left(z\partial - \bar{z}\bar{\partial}\right)X^{\mu}(z,\bar{z})\big|_{z=\bar{z}} = 0, \tag{17}$$

where $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$.

(c) For open strings with Neumann boundary conditions, the worldsheet fields have the following operator product expansion:

$$X(z,\bar{z})X(w,\bar{w}) = G(z,\bar{z};w,\bar{w}) + ...,$$
(18)

where we take the target space to be one-dimensional for simplicity,

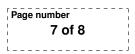
$$G(z, \bar{z}; w, \bar{w}) = -\frac{\alpha'}{2} \ln|z - w|^2 - \frac{\alpha'}{2} \ln|z - \bar{w}|^2, \qquad (19)$$

and "..." denote non-singular terms. Show that $G(z, \overline{z}; w, \overline{w})$ satisfies (17).

- (d) Using (18), compute the singular part of the operator product expansion of $\partial X(z)$ with : $e^{ipX(w,\bar{w})}$:.
- (e) Consider the vertex operator

$$V = \int_{\partial \mathcal{M}} ds \hat{V}(s), \qquad (20)$$

where the integral is over the boundary of the worldsheet, which we parameterise by the variable s, and $\hat{V}(s) = :e^{ipX(w,\bar{w})}:|_{w=\bar{w}=s}$. What must the mass of this state be in order for the vertex operator to be well-defined? What is the physical interpretation of this state? Hint: deduce the conformal weight of this state from the operator product expansion of $T(z) = -\frac{1}{\alpha'}: \partial X(z)\partial X(z):$ with $:e^{ipX(w,\bar{w})}:$ restricted to the boundary.





 $\mathbf{Q7}$ The Hamiltonian for a quantum mechanical system takes the form

$$\hat{H} = \frac{1}{2} \{ \hat{Q}, \hat{Q}^\dagger \} \,, \label{eq:Hamiltonian}$$

with \hat{Q} a fermionic operator such that $\hat{Q}^2 = (\hat{Q}^{\dagger})^2 = 0$.

- (a) Check that the operators \hat{Q} and \hat{Q}^{\dagger} are symmetries of the system, i.e. they commute with the Hamiltonian.
- (b) Show that the only possible eigenvalue for \hat{Q} and \hat{Q}^{\dagger} is zero.
- (c) Consider a state $|\psi\rangle$ such that $\hat{Q} |\psi\rangle = 0$ and $\hat{H} |\psi\rangle = E |\psi\rangle$ with $E \neq 0$. Find a second state $|\chi\rangle$ such that $\hat{H} |\chi\rangle = E |\chi\rangle$. Why does this argument fail when E = 0?

Q8 Consider the fermionic zero-modes in the Ramond sector of the RNS superstring given by the operators \hat{d}_0^i with $i \in \{2, ..., 9\}$ which obey the algebra

$$\{\hat{d}_0^i, \hat{d}_0^j\} = \delta^{ij}$$
.

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Upon rescaling $\gamma^i \doteq \sqrt{2} \hat{d}_0^i$ we obtain the Clifford algebra in 8 Euclidean dimensions:

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \,,$$

and then define

$$A^a_{\pm} \doteq \frac{1}{2} (\gamma^{2a} \pm i \gamma^{2a+1}) \,,$$

with $a \in \{1, .., 4\}$.

(a) Show that the operators A^a_{\pm} satisfy

$$\{A^a_-,A^b_+\}=\delta^{ab}\qquad;\qquad \{A^a_+,A^b_+\}=\{A^a_-,A^b_-\}=0\,.$$

(b) Define the chirality operator

$$\Gamma = \prod_{i=2}^{9} \gamma^{i} = \gamma^{2} \cdots \gamma^{9} \,,$$

and show that $\{\Gamma, A_{+}^{a}\} = 0$ for all $a \in \{1, ..., 4\}$.

(c) Define the vacuum state $|0\rangle_{\mathsf{R}}$ by imposing $A^{a}_{-}|0\rangle_{\mathsf{R}} = 0$ for all $a \in \{1, ..., 4\}$. Show that $\Gamma |0\rangle_{\mathsf{R}} = + |0\rangle_{\mathsf{R}}$ and that on the vector subspace

$$V_{8c} = \operatorname{span}\{A_{+}^{a_{1}}|0\rangle_{\mathsf{R}}, \ A_{+}^{a_{1}}A_{+}^{a_{2}}A_{+}^{a_{3}}|0\rangle_{\mathsf{R}}\}$$

the operator Γ acts as $-\hat{I}$ with \hat{I} the identity operator.