

EXAMINATION PAPER

Examination Session: May/June

Year: 2025

Exam Code:

MATH4281-WE01

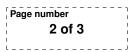
Title:

Topics in Combinatorics IV

Time:	3 hours	
Additional Material provided:		
Matariala Davraittadu		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators
Calculators r ermitted.	INO	is forbidden.

Instructions to Candidates:	Answer all questions.	
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.	
	Write your answer in the white-covered answer booklet with barcodes.	
	Begin your answer to each question on a new page.	

Revision:



SECTION A

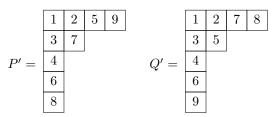
- **Q1** (a) Let $n \ge 2$ be an integer. Compute the number of Dyck paths of length 2n whose first peak is at height 2.
 - (b) Denote by $p_k(n)$ the number of Young diagrams $\lambda \vdash n$ with exactly k rows. Show that

$$p_k(n) = p_k(n-k) + p_{k-1}(n-k) + p_{k-2}(n-2)$$

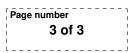
for all n > k and k > 2.

Hint: Partition the $p_k(n)$ diagrams according to the number of boxes in rows k-1 and k.

- Q2 (a) Let $w = 468593712 \in S_9$. Apply the Robinson–Schensted–Knuth (RSK) algorithm to compute the insertion and recording tableaux P and Q.
 - (b) Let (P',Q') be a pair of standard Young tableaux of shape $\lambda = (4,2,1,1,1) \vdash 9$, where



- (i) Find $w' \in S_9$ which is taken to the pair (P', Q') by the RSK algorithm.
- (ii) Find $w'' \in S_9$ which is taken to the pair (Q', P') by the RSK algorithm.
- **Q3** Let Δ be a root system of type A_3 .
 - (a) Find the Coxeter number of Δ .
 - (b) Let W be the Weyl group of Δ , i.e. $W = \langle s_1, s_2, s_3 | s_i^2, (s_1s_2)^3, (s_2s_3)^3, (s_1s_3)^2 \rangle$. Show that the subgroup of W generated by s_2 and s_1s_3 is dihedral and find its order.
- **Q4** Let Δ be a root system of type B_4 , and denote by Δ_l the set of long roots of Δ .
 - (a) Show that Δ_l is a root system and find its type.
 - (b) Let W be the Weyl group of Δ_l . Show that any two reflections in W are conjugated to each other, i.e., for any two reflections r_1 and r_2 there exists $g \in W$ such that $r_1 = gr_2g^{-1}$.



SECTION B

- **Q5** Given a permutation $w = w_1 \cdots w_n \in S_n$, recall that an *inversion* of w is a pair of integers $i, j \in \{1, \ldots, n\}$ with i < j and $w_i > w_j$. Let inv(w) denote the number of inversions of w.
 - (a) Calculate inv(w) for each permutation $w \in S_3$, and for n = 3 show that the generating function $f_{inv}(x) := \sum_{w \in S_n} x^{inv(w)}$ is of the form

$$f_{\rm inv}(x) = \prod_{k=0}^2 g_k(x),$$

where g_k is a polynomial of degree k that you should determine.

- (b) State and prove a formula for $f_{inv}(x)$ that holds for general $n \in \mathbb{N}$.
- (c) Define the *Major index* maj(w) of a permutation $w \in S_n$, and show that maj and inv are equidistributed permutation statistics.
- **Q6** (a) Carefully define the Young lattice \mathbb{Y} , and prove that it is indeed a lattice. Determine whether \mathbb{Y} is a distributive lattice.
 - (b) Fix the Young diagram $\lambda = (3, 1) \vdash 4$, and let P_{λ} be the poset on boxes (i, j) of λ with $(i, j) \leq_{P_{\lambda}} (k, l)$ if and only if $i \leq k$ and $j \leq l$. Draw the Hasse diagram of P_{λ} and its poset of order ideals $J(P_{\lambda})$.
 - (c) Prove that for any Young diagram μ the number of saturated chains in \mathbb{Y} from \emptyset to μ equals the number of standard Young tableaux of shape μ .
- **Q7** Let (G, S) be a Coxeter system, $G = \langle S | s_i^2, (s_i s_j)^{m_{ij}} \rangle$, and let $T \subset S$. Define G_T to be the subgroup of G generated by elements of T. (We call G_T a standard parabolic subgroup of G.)
 - (a) Let $w = s_1 \dots s_k$ be a word, with all $s_i \in T$. Show that for any *M*-reduction $w \to w_0$ all words obtained during the procedure belong to G_T .
 - (b) Let $\Gamma = \langle T \mid s_i^2, (s_i s_j)^{m_{ij}} \rangle$. Define a homomorphism $\phi : \Gamma \to G$ by $\phi(s_i) = s_i$. Show that ker ϕ is trivial.
 - (c) Show that (G_T, T) is a Coxeter system.

Q8 Let Δ be a root system of type C_3 .

- (a) Compute the Coxeter number of Δ and the exponents of the Weyl group of Δ .
- (b) Let P be the root poset of Δ . Draw the Hasse diagram of P.
- (c) Draw the Hasse diagram of the poset of order ideals of P. Identify join-irreducible elements.