

## EXAMINATION PAPER

Examination Session: May/June

Year: 2025

Exam Code:

MATH4281-WE01

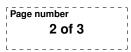
### Title:

# Topics in Combinatorics IV

Time:	3 hours	
Additional Material provided:		
Matariala Davraittadu		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators
Calculators r ermitted.	INO	is forbidden.

Instructions to Candidates:	Answer all questions.	
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.	
	Write your answer in the white-covered answer booklet with barcodes.	
	Begin your answer to each question on a new page.	

Revision:



#### SECTION A

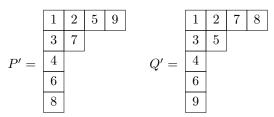
- **Q1** (a) Let  $n \ge 2$  be an integer. Compute the number of Dyck paths of length 2n whose first peak is at height 2.
  - (b) Denote by  $p_k(n)$  the number of Young diagrams  $\lambda \vdash n$  with exactly k rows. Show that

$$p_k(n) = p_k(n-k) + p_{k-1}(n-k) + p_{k-2}(n-2)$$

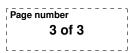
for all n > k and k > 2.

*Hint:* Partition the  $p_k(n)$  diagrams according to the number of boxes in rows k-1 and k.

- Q2 (a) Let  $w = 468593712 \in S_9$ . Apply the Robinson–Schensted–Knuth (RSK) algorithm to compute the insertion and recording tableaux P and Q.
  - (b) Let (P',Q') be a pair of standard Young tableaux of shape  $\lambda = (4,2,1,1,1) \vdash 9$ , where



- (i) Find  $w' \in S_9$  which is taken to the pair (P', Q') by the RSK algorithm.
- (ii) Find  $w'' \in S_9$  which is taken to the pair (Q', P') by the RSK algorithm.
- **Q3** Let  $\Delta$  be a root system of type  $A_3$ .
  - (a) Find the Coxeter number of  $\Delta$ .
  - (b) Let W be the Weyl group of  $\Delta$ , i.e.  $W = \langle s_1, s_2, s_3 | s_i^2, (s_1s_2)^3, (s_2s_3)^3, (s_1s_3)^2 \rangle$ . Show that the subgroup of W generated by  $s_2$  and  $s_1s_3$  is dihedral and find its order.
- **Q4** Let  $\Delta$  be a root system of type  $B_4$ , and denote by  $\Delta_l$  the set of long roots of  $\Delta$ .
  - (a) Show that  $\Delta_l$  is a root system and find its type.
  - (b) Let W be the Weyl group of  $\Delta_l$ . Show that any two reflections in W are conjugated to each other, i.e., for any two reflections  $r_1$  and  $r_2$  there exists  $g \in W$  such that  $r_1 = gr_2g^{-1}$ .



### SECTION B

- **Q5** Given a permutation  $w = w_1 \cdots w_n \in S_n$ , recall that an *inversion* of w is a pair of integers  $i, j \in \{1, \ldots, n\}$  with i < j and  $w_i > w_j$ . Let inv(w) denote the number of inversions of w.
  - (a) Calculate inv(w) for each permutation  $w \in S_3$ , and for n = 3 show that the generating function  $f_{inv}(x) := \sum_{w \in S_n} x^{inv(w)}$  is of the form

$$f_{\rm inv}(x) = \prod_{k=0}^2 g_k(x),$$

where  $g_k$  is a polynomial of degree k that you should determine.

- (b) State and prove a formula for  $f_{inv}(x)$  that holds for general  $n \in \mathbb{N}$ .
- (c) Define the *Major index* maj(w) of a permutation  $w \in S_n$ , and show that maj and inv are equidistributed permutation statistics.
- **Q6** (a) Carefully define the Young lattice  $\mathbb{Y}$ , and prove that it is indeed a lattice. Determine whether  $\mathbb{Y}$  is a distributive lattice.
  - (b) Fix the Young diagram  $\lambda = (3, 1) \vdash 4$ , and let  $P_{\lambda}$  be the poset on boxes (i, j) of  $\lambda$  with  $(i, j) \leq_{P_{\lambda}} (k, l)$  if and only if  $i \leq k$  and  $j \leq l$ . Draw the Hasse diagram of  $P_{\lambda}$  and its poset of order ideals  $J(P_{\lambda})$ .
  - (c) Prove that for any Young diagram  $\mu$  the number of saturated chains in  $\mathbb{Y}$  from  $\emptyset$  to  $\mu$  equals the number of standard Young tableaux of shape  $\mu$ .
- **Q7** Let (G, S) be a Coxeter system,  $G = \langle S | s_i^2, (s_i s_j)^{m_{ij}} \rangle$ , and let  $T \subset S$ . Define  $G_T$  to be the subgroup of G generated by elements of T. (We call  $G_T$  a standard parabolic subgroup of G.)
  - (a) Let  $w = s_1 \dots s_k$  be a word, with all  $s_i \in T$ . Show that for any *M*-reduction  $w \to w_0$  all words obtained during the procedure belong to  $G_T$ .
  - (b) Let  $\Gamma = \langle T \mid s_i^2, (s_i s_j)^{m_{ij}} \rangle$ . Define a homomorphism  $\phi : \Gamma \to G$  by  $\phi(s_i) = s_i$ . Show that ker  $\phi$  is trivial.
  - (c) Show that  $(G_T, T)$  is a Coxeter system.

**Q8** Let  $\Delta$  be a root system of type  $C_3$ .

- (a) Compute the Coxeter number of  $\Delta$  and the exponents of the Weyl group of  $\Delta$ .
- (b) Let P be the root poset of  $\Delta$ . Draw the Hasse diagram of P.
- (c) Draw the Hasse diagram of the poset of order ideals of P. Identify join-irreducible elements.