Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH42920-WE01

Title:
Functional Analysis and Applications V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Let X, Y be infinite-dimensional Banach spaces and let $T : X \to Y$ be a linear operator.

(a) Show that T is injective if and only if for every linearly independent set $\mathcal{B} \subset X$ the image set $T(\mathcal{B}) \subset Y$ is linearly independent.

(b) Show that if T is bijective and $\mathcal{B} \subset X$ is a Hamel basis of X, then the image set $T(\mathcal{B}) \subset Y$ is a Hamel basis of Y.

Q2 Let $1 \leq p \leq \infty$. For $k \in \mathbb{N}$ define $f_k : \ell^p(\mathbb{N}) \to \mathbb{C}$ by

$$f_k(\boldsymbol{x}) = x_k, \quad \boldsymbol{x} = (x_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N}).$$

(a) Show that $f_k \in \ell^p(\mathbb{N})^*$ for every $k \in \mathbb{N}$ and every $1 \leq p \leq \infty$.

(b) For which $1 \leq p \leq \infty$ is the sequence $(f_k)_{k\in\mathbb{N}}$ weakly-$*$ convergent in $\ell^p(\mathbb{N})^*$? For p for which we have weak-$*$ convergence, what is the limit? Justify your answers.

Q3 Let H be a Hilbert space and let $T \in C(H)$ be densely defined.

(a) State the definitions of the point spectrum $\sigma_p(T)$, the continuous spectrum $\sigma_c(T)$ and the residual spectrum $\sigma_r(T)$.

(b) With T^* denoting the (Hilbert space) adjoint operator, show that if $\lambda \in \sigma_r(T)$ then $\overline{\lambda} \in \sigma_p(T^*)$.

Q4 (a) Let X be a Banach space and $T \in C(X)$. Show that if $(T - \lambda_0)^{-1}$ is compact for some $\lambda_0 \in \rho(T)$ then T has compact resolvent, i.e. $(T - \lambda)^{-1}$ is compact for all $\lambda \in \rho(T)$.

(b) Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by

$$T\boldsymbol{x} = \left(\frac{n}{n+1} x_n\right)_{n\in\mathbb{N}}, \quad \boldsymbol{x} = (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}).$$

Show that T does not have compact resolvent.

SECTION B

Q5 For $k \in \mathbb{N}$, let $\boldsymbol{e}_k$ denote the k-th standard basis element of $\ell^2(\mathbb{N})$, i.e. $(\boldsymbol{e}_k)_n = \delta_{kn}$, and define $T_k : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$T_k \boldsymbol{x} = k x_k \boldsymbol{e}_k, \quad \boldsymbol{x} = (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}).$$

(a) Show that for every $k \in \mathbb{N}$ the operator T_k is bounded and find its operator norm $\|T_k\|$.

(b) Check for each of the following cases whether there exists a bounded linear operator $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ satisfying the respective convergence:

(i) $T_k \boldsymbol{x} \xrightarrow{w} T\boldsymbol{x}$ as $k \to \infty$ for every $\boldsymbol{x} \in \ell^2(\mathbb{N})$.

(ii) $T_k \boldsymbol{x} \to T\boldsymbol{x}$ as $k \to \infty$ for every $\boldsymbol{x} \in \ell^2(\mathbb{N})$.

(iii) $\|T_k - T\| \to 0$ as $k \to \infty$.

Justify your answers.

Q6 (a) Show that the set

$$\big\{ f \in L^2(0,\infty) : \exists x_0 \geq 0 \text{ such that } \forall\, x \geq x_0 : \ f(x) = 0 \big\}$$

is dense in $L^2(0,\infty)$.

(b) Define $T : L^2(0,\infty) \to L^2(0,\infty)$ by

$$(Tf)(x) = m(x) f(x), \quad f \in L^2(0,\infty),$$

with function

$$m(x) = \sum_{n=1}^{\infty} \frac{1}{n} \chi_{[n-1,n)}(x)$$

where $\chi_{[n-1,n)}(x)$ denotes the characteristic function of the interval $[n-1,n)$. Show that the range $\mathcal{R}(T) \subset L^2(0,\infty)$ is dense but $\mathcal{R}(T) \neq L^2(0,\infty)$.

Q7 Define $T : \mathcal{D}(T) \subset \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$T\boldsymbol{x} = \left(\sum_{n=k}^{\infty} x_n \right)_{k\in\mathbb{N}}, \quad \boldsymbol{x} = (x_n)_{n\in\mathbb{N}} \in \mathcal{D}(T)$$

on the operator domain

$$\mathcal{D}(T) = \left\{ \boldsymbol{x} = (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) : \sum_{k=1}^{\infty} \left| \sum_{n=k}^{\infty} x_n \right|^2 < \infty \right\}.$$

Define

$$\Phi := \left\{ \boldsymbol{y} = (y_n)_{n\in\mathbb{N}} \in \mathcal{D}(T) : \sum_{n=1}^{\infty} y_n = 0 \right\}.$$

(a) Show that the (Hilbert space) adjoint operator satisfies $\Phi \subset \mathcal{D}(T^*)$ and

$$T^*\boldsymbol{y} = -T\boldsymbol{y} + \boldsymbol{y}, \quad \boldsymbol{y} \in \Phi.$$

(b) Show that the operator $S = \mathrm{i}(T^* - \frac{1}{2})$ on $\mathcal{D}(S) = \Phi$ is a symmetric operator. You may use without proof that $\Phi \subset \ell^2(\mathbb{N})$ is dense.

Q8 Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $T : H \to H$ be a bounded linear operator. Define

$$B(\varphi, u) = \langle T\varphi, Tu \rangle, \quad \varphi, u \in H.$$

(a) Show that B is a bounded sesquilinear form.

(b) Show that B is coercive if the point 0 is in the resolvent set $\rho(T)$.

(c) Assume that $0 \in \rho(T)$ and let $f \in H^*$. Show that there exists a unique $u \in H$ such that

$$\forall\, \varphi \in H : \; B(\varphi, u) = f(\varphi). \tag{1}$$

For $f = \langle \cdot, x \rangle$ with a fixed $x \in H$, find an explicit expression for the unique solution u of (1).