Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH43020-WE01

Title:
Stochastic Processes V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Let X, Y be two non-negative random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[X], \mathbb{E}[Y] < \infty$.

(a) State the definition of $X \leq_{\text{st}} Y$.

(b) Suppose $X \leq_{\text{st}} Y$. Show that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.

(c) Does $\mathbb{E}[X] \leq \mathbb{E}[Y]$ imply $X \leq_{\text{st}} Y$? Provide a proof if this is true, or a counter-example otherwise.

Q2 Let $(N(t))_{t\geq 0}$ be a Poisson process with intensity λ. Assume that $N(0) = 0$.

(a) Write down a formula for the distribution of $N(t)$ for fixed $t > 0$, that is, what is the value of $\mathbb{P}[N(t) = k]$ for integers $k \geq 0$?

(b) Let $s, t > 0$ with $0 < s < t$. Explain why for $m > n$

$$\mathbb{P}[N(s) = m \mid N(t) = n] = 0,$$

and for $0 \leq m \leq n$ derive a formula for

$$\mathbb{P}[N(s) = m \mid N(t) = n].$$

Q3 (a) State the definition of a martingale sequence. Make sure to state all probabilistic objects and conditions involved in the definition.

(b) Let $X_1, X_2, \ldots$ be a sequence of independent random variables with

$$\mathbb{P}[X_n = i] = \begin{cases} e^{-n} & \text{if } i = 0 \\ 1 - 2e^{-n} & \text{if } i = 1 \\ e^{-n} & \text{if } i = 2 \end{cases} \qquad \text{for } n \geq 1.$$

Let $M_n = \prod_{i=1}^{n} X_i$. Specify a filtration and show that M_n is a martingale with respect to this filtration.

SECTION B

Q4 (a) State the definition of the total variation distance $d_{\text{TV}}(U,V)$ between two real-valued random variables U, V.

(b) State and prove the triangle inequality for total variation distance.

(c) What is the largest value that the total variation distance may take? Prove this best upper bound and show that this can be attained with suitable examples of random variables.

(d) Let $(X_n)_{n\geq 0}$ be a branching process with $X_0 = 1$ and $\varphi_X(s) := \mathbb{E}[s^{X_1}] = \frac{1}{4}(1+s+s^2+s^3)$. Find the extinction probability $\rho_X := \lim_{n\to\infty} \mathbb{P}(X_n = 0)$.

(e) Continuing from the previous part, let $(Y_n)_{n\geq 0}$ be another branching process with $Y_0 = 10$ and

$$\varphi_Y(s) := \mathbb{E}[s^{Y_2}|Y_1 = 1] = \frac{1}{4}(1+s^3) + \frac{1}{2}e^{\frac{1}{2}(s-1)}.$$

Evaluate $\lim_{n\to\infty} d_{\text{TV}}(X_n, Y_n)$.
(Hint: what is the mean number of offspring of each individual in $(Y_n)_{n\geq 0}$?)

(You may use any results concerning branching processes provided that they are stated carefully.)

Q5 (a) What does it mean for a real-valued random variable X to be arithmetic? Give an example of a 3-arithmetic random variable X satisfying $\text{Var}(X) = 2$.

(b) Let $d > 0$. State the Blackwell renewal theorem for d-arithmetic distributions.

(c) You are generating a random sequence of alphabets in the following way:

- Alphabets are picked independently according to the same distribution.
- The distribution of each alphabet is as follows: with probability $\frac{1}{4}$, choose a vowel $\{\text{A}, \text{E}, \text{I}, \text{O}, \text{U}\}$ uniformly at random; otherwise (i.e. with probability $\frac{3}{4}$), choose one of the 21 consonants uniformly at random.

Let τ be the time for the first appearance of MEME in consecutive order. Find $\mathbb{E}[\tau]$ by applying results for suitable renewal processes carefully.

Q6 Let $X_1 = (X_1(t))_{t\geq 0}$ and $X_2 = (X_2(t))_{t\geq 0}$ be continuous time Markov processes on $\{0,1\}$, with common generator

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} \qquad \alpha, \beta > 0.$$

Assume that the processes X_1 and X_2 are independent, that is $X_1(u)$ and $X_2(v)$ are independent for all $u, v \geq 0$. Let $X(t) = X_1(t) + X_2(t)$ for all $t \geq 0$.

(a) Show that $X = (X(t))_{t\geq 0}$ is a continuous time Markov process on $\{0,1,2\}$ and find its generator.

(b) Show that X is an irreducible Markov process.

(c) Find the stationary distribution of X.

(d) Find

$$\lim_{t\to\infty} \mathbb{P}[X(t) = 1 \,|\, X(0) = 0].$$

Q7 At time $n = 0$, an urn contains 1 blue ball and 1 red ball. At each time $n = 1, 2, 3, \ldots$, a ball is chosen at random from the urn and returned to the urn together with a new ball of the same colour. Just after time n, there are $n + 2$ balls in the urn of which B_n are blue and R_n are red. Let

$$M_n = \frac{B_n}{B_n + R_n} = \frac{B_n}{n+2}$$

be the proportion of blue balls in the urn after time n.

(a) Show that M_n is a martingale sequence. You should specify the filtration.

(b) Show that for an integer k with $1 \leq k \leq n+1$,

$$\mathbb{P}[B_n = k] = \frac{1}{n+1}.$$

(c) Let T be the first time a blue ball is added to the urn. Compute

$$\mathbb{E}\left[\frac{1}{T+2}\right]$$

carefully stating any theorems you use from the course.

SECTION C

Q8 Let $(Z_n^1, Z_n^2)_{n\geq 0}$ be a two-type time-homogeneous branching process with offspring distribution satisfying

$$f^1(s_1, s_2) := \mathbb{E}\left[s_1^{Z_1^1} s_2^{Z_1^2} \middle| (Z_0^1, Z_0^2) = (1,0)\right] = \frac{1}{4}s_1 s_2 + a e^{M(s_2-1)} + b$$

$$\text{and} \quad f^2(s_1, s_2) := \mathbb{E}\left[s_1^{Z_1^1} s_2^{Z_1^2} \middle| (Z_0^1, Z_0^2) = (0,1)\right] = \frac{1}{64}(1+s_1)^4 + \frac{1}{8}s_1 s_2^2 + \frac{5}{8}$$

where $M \geq 0$ and $a, b \geq 0$ are some suitable constants such that $f^1(s_1, s_2)$ is a valid generating function. Determine the condition on (M, a) under which the process becomes extinct with probability 1.