

EXAMINATION PAPER

Examination Session: May/June

2025

Year:

Exam Code:

MATH43320-WE01

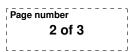
Title:

Ergodic Theory V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions.	
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.	
	Write your answer in the white-covered answer booklet with barcodes.	
	Begin your answer to each question on a new page.	

Revision:



SECTION A

- **Q1** Throughout this problem, (X, T) and (Z, S) are topological dynamical systems.
 - (a) Give two alternative (but equivalent) characterisations of what it means for (X,T) to be **minimal**.
 - (b) Suppose the product system $(X \times Z, T \times S)$ is minimal. Show that (X, T) and (Z, S) are also minimal.
 - (c) Now suppose (X, T) and (Z, S) are minimal. Is it necessarily true that the product $(X \times Z, T \times S)$ is also minimal? Justify your answer.
- Q2 (a) State Poincare's recurrence theorem.
 - (b) Prove Poincare's recurrence theorem.
- **Q3** Let $n \in \mathbb{N}_{\geq 2}$. For each $x = (x_k)_{k \in \mathbb{N}} \in \{1, \ldots, n\}^{\mathbb{N}}$ (that is, for each sequence with entries $x_k \in \{1, \ldots, n\}$) and each $j \in \{1, \ldots, n\}$, we define

$$freq_j(x) := \lim_{N \to \infty} \frac{1}{N} \# \{ 0 \le k \le N - 1 : x_k = j \}$$

if this limit exists.

(a) Let $A \in \{0,1\}^{n \times n}$ be an irreducible adjacency matrix and (Σ_A^+, σ) the corresponding subshift of finite type. Suppose P is a stochastic matrix which is compatible with A and let μ_P be the associated Markov measure. Show that for each $i \in \{1, \dots, n\}$ and μ_P almost all $x \in \Sigma^+$ free (x) exists

Show that for each $j \in \{1, ..., n\}$ and μ_P -almost all $x \in \Sigma_A^+$, freq_j(x) exists, and compute its value.

(b) Given any $\alpha \in (0,1)$, show that there is an uncountable set of points $x \in \{1, \ldots, n\}^{\mathbb{N}}$ such that $\operatorname{freq}_1(x) = \alpha$.

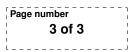
Hint: Pick some positive row vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$ with $\pi_1 = \alpha$ and $\pi_1 + \pi_2 + \ldots + \pi_n = 1$. Find a stochastic matrix P with $\boldsymbol{\pi} P = \boldsymbol{\pi}$ and consider the Markov measure μ_P for an appropriate subshift of finite type. What is freq₁(x) for μ_P -almost all x in this subshift? Show that $\mu_P(\{x\}) = 0$ for every singleton set $\{x\}$ and conclude that only uncountable sets have positive measure.

Q4 Let $f: [0,1] \to [0,1]$ be a strictly increasing continuous map on the unit interval with exactly two fixed points 0 and 1. Throughout this problem, you may use without proof that each invariant measure μ of ([0,1], f) is of the form

$$\mu = \lambda \cdot \delta_0 + (1 - \lambda) \cdot \delta_1$$

for some $\lambda \in [0, 1]$. Here δ_0 and δ_1 denote the δ -measure at 0 and 1, respectively.

- (a) Which are the ergodic measures of ([0, 1], f)? Justify your answer.
- (b) For each invariant measure, compute the metric entropy.
- (c) What is the topological entropy of ([0, 1], f)? Justify your answer.



SECTION B

- Q5 (a) Give an example of a topological dynamical system on $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ which has no fixed points. Justify your answer.
 - (b) Show that every topological dynamical system on the unit interval [0, 1] has a fixed point.
 - (c) Suppose (X, f) and (Y, g) are topologically conjugate dynamical systems. Show that if (X, f) has a fixed point, then (Y, g) has a fixed point and vice versa.
 - (d) Using (a), (b), and (c), show that \mathbb{T}^1 is not homeomorphic to [0, 1].
- **Q6** In the following, (X, T) is a topological dynamical system. Recall that we call (X, T) topologically mixing if for each pair of open sets $U, V \subseteq X$ there is $n \in \mathbb{N}$ such that for all $N \ge n$, we have $T^N(U) \cap V \ne \emptyset$.
 - (a) Let μ be an invariant measure of (X, T). State the definition of when we call the measure preserving system (X, \mathcal{B}, μ, T) mixing.
 - (b) Suppose μ assigns a positive value to each non-empty open set, that is, for every open set $U \neq \emptyset$, we have $\mu(U) > 0$. Suppose further that (X, \mathcal{B}, μ, T) is mixing. Show that (X, T) is topologically mixing.
 - (c) True or False: If (X, T) is topologically mixing and μ is an invariant measure of (X, T) which assigns a positive value to each non-empty open set, then (X, T) is mixing with respect to μ . Justify your answer.
- **Q7** (a) Suppose X is a compact metric space. State the definition of a **positively** expansive continuous map $T: X \to X$.
 - (b) Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be the doubling map, that is, T[x] = [2x] for all $[x] \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Show that $\frac{1}{4}$ is an expansivity constant for T. Hint: Without proof, you may use the fact that for all $[x], [y] \in \mathbb{T}^1$,

either
$$d_{\mathbb{T}^1}([x], [y]) > \frac{1}{4}$$
 or $d_{\mathbb{T}^1}(T[x], T[y]) = 2 \cdot d_{\mathbb{T}^1}([x], [y]).$

- (c) Show that a positively expansive continuous map on a compact metric space X has finitely many fixed points.
- **Q8** Let $n \in \mathbb{N}_{\geq 2}$. Recall that the **one-sided full-shift (on** n **symbols)** is the topological dynamical system $(\{1, 2, \ldots, n\}^{\mathbb{N}}, \sigma)$ where $\{1, 2, \ldots, n\}^{\mathbb{N}}$ is the collection of all one-sided infinite sequences $(x_k)_{k \in \mathbb{N}}$ with entries $x_k \in \{1, 2, \ldots, n\}$ and σ is the left shift.
 - (a) Show that the one-sided full-shift is positively expansive.
 - (b) Compute the topological entropy h of the one-sided full-shift on n symbols.
 - (c) Is there a subshift of $(\{1, 2, ..., n\}^{\mathbb{N}}, \sigma)$ whose topological entropy is bigger than the topological entropy h of the one-sided full-shift on n symbols? Justify your answer.