

## EXAMINATION PAPER

Examination Session: May/June

2025

Year:

Exam Code:

MATH4341-WE01

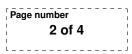
### Title:

# Spatio-Temporal Statistics

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
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Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions.	
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.	
	Write your answer in the white-covered answer booklet with barcodes.	
	Begin your answer to each question on a new page.	

**Revision:** 



### SECTION A

Q1 (a) Consider the semi-variogram

$$\gamma \left( h \right) = \begin{cases} 0 & \text{if } h = 0\\ \alpha + \beta \left( \frac{3|h|}{2} - \frac{|h|^3}{2} \right) & \text{if } |h| \in (0, 1)\\ \alpha + \beta & \text{if } |h| \ge 1 \end{cases}$$

for  $\alpha > 0$  and  $\beta > 0$ . What are the sill, range, nugget, and partial sill for this covariance model? Justify your answers.

- (b) Let  $\{Z(s) : s \in S\}$  with  $S = (-\pi, \pi)$  be a Gaussian process with mean function  $\mu(s) = m$  where  $m \in \mathbb{R}$  is a constant and covariance function  $c(s,t) = \frac{1}{2}(|s| + |t| |t s|)$  for  $s, t \in S$ . Report whether or not the stochastic process  $Z(\cdot)$  is weakly stationary, intrinsically stationary, continuous, and everywhere differentiable. Justify your answer.
- (c) Let B be a  $n \times n$  symmetric matrix with zero-valued diagonal elements (namely  $[B]_{s,s} = 0$  for s = 1, ..., n) and such that (I B) is positive definite, where I denotes the identity matrix. Consider the conditional autoregression Gaussian model on a finite family  $S = \{1, ..., n\}, n > 1$ , of sites defined by Gaussian local characteristics, with

$$\mathbb{E}\left(Z_t|Z_{\mathcal{S}\setminus t}\right) = \mu + \sum_{s\neq t} \left[B\right]_{s,t} \left(Z_s - \mu\right)$$

and  $\operatorname{Var}(Z_t|Z_{S\setminus t}) = 1$  for  $t \in S$  for some unknown parameter  $\mu \in \mathbb{R}$ . Show that the joint distribution of  $Z = (Z_1, ..., Z_n)^{\top}$  is Gaussian with mean  $\mu \underline{1}$ and covariance matrix  $(I - B)^{-1}$ . We denote the column vector of ones as  $\underline{1} = (1, ..., 1)^{\top}$ .

 $\mathbf{Q2}$  Consider the model

$$X_{t} = \frac{5}{6}X_{t-1} - \frac{1}{6}X_{t-2} + \varepsilon_{t} - \frac{1}{2}\varepsilon_{t-1} - \frac{1}{4}\varepsilon_{t-2} + \frac{1}{8}\varepsilon_{t-3}, \ \varepsilon_{t} \sim N(0, \sigma^{2}), \ t \in \mathbb{Z},$$

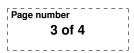
of mixed auto-regressive moving average form of order ARMA(2, 3).

(a) Show that this model contains a parameter-redundancy allowing simplification to a model of reduced order, and that this simplified model can be written in the form

$$X_t = \frac{1}{3}X_{t-1} + \varepsilon_t - \frac{1}{4}\varepsilon_{t-2}.$$

Identify the order of this reduced model.

- (b) Check that the simplified model is stationary, and then compute the stationary variance of the process (in terms of  $\sigma > 0$ ).
- (c) Compute the spectral density function,  $S(\nu)$ , of this process as a function of  $\nu \in [0, \frac{1}{2}]$  and  $\sigma$ . Your final expression should not involve the imaginary unit, i.





#### SECTION B

**Q3** Consider a set of random fields  $\left\{ \left( Z_{j}^{\left(u\right)}\left(s\right):s\in\mathcal{S}\right);j=1,...,k;u=1,...,k\right\}$  with

$$Z_{j}^{(u)}(s) = \sum_{p=1}^{k} a_{j,p}^{(u)} w_{p}^{(u)}(s) ,$$

where  $\left\{w_p^{(u)}(s)\right\}$  are intrinsic random fields and  $\left\{a_{j,p}^{(u)}\right\}$  are known constants. Let  $\tilde{\gamma}_{i,j}^{(u)}(h)$  be the cross-variogram function of  $Z_i^{(u)}(s)$  and  $Z_j^{(u)}(s)$  for u = 1, ..., k.

- (a) Write the definition of the cross-variogram function  $\tilde{\gamma}_{i,j}^{(u)}(h)$  of  $Z_i^{(u)}(s)$  and  $Z_j^{(u)}(s)$  for u = 1, ..., k
- (b) Assume that

$$E\left(w_{p}^{(u)}\left(s\right)\right) = 0$$
  
$$\operatorname{Cov}\left(w_{p}^{(u)}\left(s\right), w_{q}^{(v)}\left(s+h\right)\right) = \begin{cases} \gamma_{p,q}^{(u)}\left(h\right), & u = v\\ 0 & u \neq v \end{cases}$$

for u = 1, ..., k, p = 1, ..., k and q = 1, ..., k. Show that

$$\tilde{\gamma}_{i,j}^{(u)}(h) = \sum_{p=1}^{k} a_{i,p}^{(u)} \sum_{q=1}^{k} a_{j,q}^{(u)} \gamma_{p,q}^{(u)}(h)$$

(c) Assume that

$$E\left(w_{p}^{\left(u\right)}\left(s\right)\right) = 0$$
  
Cov  $\left(w_{p}^{\left(u\right)}\left(s\right), w_{q}^{\left(v\right)}\left(s+h\right)\right) = \begin{cases} \gamma^{\left(u\right)}\left(h\right), & u = v \text{ and } p = q\\ 0 & u \neq v \text{ or } p \neq q \end{cases}$ 

for u = 1, ..., k. For u = 1, ..., k, compute the cross-variogram matrix  $\tilde{\Gamma}^{(u)}(h)$  of vector

$$\left(Z_{1}^{\left(u\right)}\left(s\right),...,Z_{k}^{\left(u\right)}\left(s\right)\right)^{\top}$$

in the form

$$\tilde{\Gamma}^{(u)}\left(h\right) = B^{(u)}\gamma^{(u)}\left(h\right)$$

and express quantities  $B^{(u)}$  as functions of matrix  $A^{(u)}$  with  $[A^{(u)}]_{i,p} = a^{(u)}_{i,p}$ .

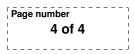
(d) Consider the assumptions in the previous part. Let  $\{(Z_j(s) : s \in S); j = 1, ..., k\}$  be a set of random fields on  $s \in S$ . Let

$$Z_{j}(s) = \mu_{j}(s) + \sum_{u=0}^{m} Z_{j}^{(u)}(s)$$

Show that the cross-variogram matrix of  $(Z(s); s \in S)$ where  $Z(s) = (Z_1(s), ..., Z_k(s))^{\top}$  is

$$\Gamma(h) = \sum_{u=0}^{m} B^{(u)} \gamma^{(u)}(h)$$

CONTINUED





Q4 Consider the constant (time-invariant) dynamic linear model

$$\begin{split} \mathbf{Y}_t &= \mathsf{F}\mathbf{X}_t + \boldsymbol{\nu}_t, \quad \boldsymbol{\nu}_t \sim N(\mathbf{0},\mathsf{V}), \\ \mathbf{X}_t &= \mathsf{G}\mathbf{X}_{t-1} + \boldsymbol{\omega}_t, \quad \boldsymbol{\omega}_t \sim N(\mathbf{0},\mathsf{W}), \end{split}$$

for *m*-dimensional observation vectors,  $\mathbf{Y}_t$ , *p*-dimensional hidden states  $\mathbf{X}_t$ , and fully specified matrices  $\mathsf{F}, \mathsf{G}, \mathsf{V}, \mathsf{W}$  (of appropriate dimensions). The model is considered for  $t = 1, 2, \ldots$ , and initialised with  $\mathbf{X}_0 \sim N(\mathbf{m}_0, \mathsf{C}_0)$ . Given *n* observations  $\mathbf{y}_{1:n} = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$ , sequential computation of the filtered distributions

$$(\mathbf{X}_t | \mathbf{Y}_{1:t} = \mathbf{y}_{1:t}) \sim N(\mathbf{m}_t, \mathsf{C}_t), \quad t = 1, \dots, n,$$

has been carried out using the Kalman filter. Interest now focuses on computing the smoothing distributions

$$(\mathbf{X}_t | \mathbf{Y}_{1:n} = \mathbf{y}_{1:n}) \sim N(\mathbf{s}_t, \mathbf{S}_t), \quad t = n, \dots, 1.$$

We begin by noting that  $\mathbf{s}_n = \mathbf{m}_n$  and  $\mathsf{S}_n = \mathsf{C}_n$ .

(a) (i) Consider the problem at time t < n, where we have already computed  $\mathbf{s}_{t+1}$ ,  $\mathsf{S}_{t+1}$ . Write down the form of the joint distribution of

$$\begin{pmatrix} \mathbf{X}_t \\ \mathbf{X}_{t+1} \\ \end{bmatrix} \mathbf{y}_{1:t}$$

(ii) Use multivariate normal conditioning to show that

$$(\mathbf{X}_t | \mathbf{X}_{t+1}, \mathbf{y}_{1:t}) \sim N\left(\mathbf{m}_t + \mathsf{L}_t [\mathbf{X}_{t+1} - \tilde{\mathbf{m}}_{t+1}], \ \mathsf{C}_t - \mathsf{L}_t \tilde{\mathsf{C}}_{t+1} \mathsf{L}_t^{\mathsf{T}}\right),$$

where  $\tilde{\mathbf{m}}_{t+1} = \mathsf{G}\mathbf{m}_t$ ,  $\tilde{\mathsf{C}}_{t+1} = \mathsf{G}\mathsf{C}_t\mathsf{G}^\mathsf{T} + \mathsf{W}$ , and  $\mathsf{L}_t = \mathsf{C}_t\mathsf{G}^\mathsf{T}\tilde{\mathsf{C}}_{t+1}^{-1}$ .

- (iii) Explain why this is also the distribution of  $(\mathbf{X}_t | \mathbf{X}_{t+1}, \mathbf{y}_{1:n})$ , and then marginalise out  $\mathbf{X}_{t+1}$  to obtain expressions for  $\mathbf{s}_t$  and  $\mathbf{S}_t$ .
- (b) Explain how you would modify the above backward smoothing procedure to instead generate an exact sample from the conditional distribution  $(\mathbf{X}_{1:n}|\mathbf{y}_{1:n})$ .
- (c) (i) Suppose that we wish to model a monthly (period 12) univariate time series using a dynamic linear model consisting of a locally constant trend and a Fourier-based seasonal effect using two harmonics. How would you structure this model? Give explicit forms for F and G, and suggest an appropriate structural form for W (though this may contain unspecified parameters).
  - (ii) Discuss briefly one approach that could be used to estimate any unspecified parameters in the model. Detailed formulas are not required.