EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH4361-WE01

Title: Ergodic Theory IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 Throughout this problem, (X, T) and (Z, S) are topological dynamical systems.

(a) Give two alternative (but equivalent) characterisations of what it means for (X, T) to be **minimal**.

(b) Suppose the product system $(X \times Z, T \times S)$ is minimal. Show that (X, T) and (Z, S) are also minimal.

(c) Now suppose (X, T) and (Z, S) are minimal. Is it necessarily true that the product $(X \times Z, T \times S)$ is also minimal? Justify your answer.

Q2 (a) State Poincare's recurrence theorem.

(b) Prove Poincare's recurrence theorem.

Q3 Let $n \in \mathbb{N}_{\geq 2}$. For each $x = (x_k)_{k \in \mathbb{N}} \in \{1, \ldots, n\}^{\mathbb{N}}$ (that is, for each sequence with entries $x_k \in \{1, \ldots, n\}$) and each $j \in \{1, \ldots, n\}$, we define

$$\operatorname{freq}_j(x) := \lim_{N \to \infty} \frac{1}{N} \#\{0 \leq k \leq N-1 : \ x_k = j\}$$

if this limit exists.

(a) Let $A \in \{0,1\}^{n \times n}$ be an irreducible adjacency matrix and (Σ_A^+, σ) the corresponding subshift of finite type. Suppose P is a stochastic matrix which is compatible with A and let μ_P be the associated Markov measure.

Show that for each $j \in \{1, \ldots, n\}$ and μ_P-almost all $x \in \Sigma_A^+$, $\operatorname{freq}_j(x)$ exists, and compute its value.

(b) Given any $\alpha \in (0,1)$, show that there is an uncountable set of points $x \in \{1, \ldots, n\}^{\mathbb{N}}$ such that $\operatorname{freq}_1(x) = \alpha$.

Hint: Pick some positive row vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ *with* $\pi_1 = \alpha$ *and* $\pi_1 + \pi_2 + \ldots + \pi_n = 1$. *Find a stochastic matrix* P *with* $\boldsymbol{\pi} P = \boldsymbol{\pi}$ *and consider the Markov measure* μ_P *for an appropriate subshift of finite type. What is* $\operatorname{freq}_1(x)$ *for* μ_P*-almost all* x *in this subshift? Show that* $\mu_P(\{x\}) = 0$ *for every singleton set* $\{x\}$ *and conclude that only uncountable sets have positive measure.*

Q4 Let $f\colon [0,1] \to [0,1]$ be a strictly increasing continuous map on the unit interval with exactly two fixed points 0 and 1. Throughout this problem, you may use without proof that each invariant measure μ of $([0,1], f)$ is of the form

$$\mu = \lambda \cdot \delta_0 + (1 - \lambda) \cdot \delta_1$$

for some $\lambda \in [0,1]$. Here δ_0 and δ_1 denote the δ-measure at 0 and 1, respectively.

(a) Which are the ergodic measures of $([0,1], f)$? Justify your answer.

(b) For each invariant measure, compute the metric entropy.

(c) What is the topological entropy of $([0,1], f)$? Justify your answer.

SECTION B

Q5 (a) Give an example of a topological dynamical system on $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ which has no fixed points. Justify your answer.

(b) Show that every topological dynamical system on the unit interval $[0, 1]$ has a fixed point.

(c) Suppose (X, f) and (Y, g) are topologically conjugate dynamical systems. Show that if (X, f) has a fixed point, then (Y, g) has a fixed point and vice versa.

(d) Using (a), (b), and (c), show that $\mathbb{T}^1$ is not homeomorphic to $[0, 1]$.

Q6 In the following, (X, T) is a topological dynamical system. Recall that we call (X, T) **topologically mixing** if for each pair of open sets $U, V \subseteq X$ there is $n \in \mathbb{N}$ such that for all $N \geq n$, we have $T^N(U) \cap V \neq \emptyset$.

(a) Let μ be an invariant measure of (X, T). State the definition of when we call the measure preserving system $(X, \mathcal{B}, \mu, T)$ **mixing**.

(b) Suppose μ assigns a positive value to each non-empty open set, that is, for every open set $U \neq \emptyset$, we have $\mu(U) > 0$. Suppose further that $(X, \mathcal{B}, \mu, T)$ is mixing. Show that (X, T) is topologically mixing.

(c) True or False: If (X, T) is topologically mixing and μ is an invariant measure of (X, T) which assigns a positive value to each non-empty open set, then (X, T) is mixing with respect to μ. Justify your answer.

Q7 (a) Suppose X is a compact metric space. State the definition of a **positively expansive** continuous map $T : X \to X$.

(b) Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be the doubling map, that is, $T[x] = [2x]$ for all $[x] \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Show that $\frac{1}{4}$ is an expansivity constant for T.
Hint: Without proof, you may use the fact that for all $[x], [y] \in \mathbb{T}^1$,

$$\textit{either} \quad d_{\mathbb{T}^1}([x], [y]) > \frac{1}{4} \quad \textit{or} \quad d_{\mathbb{T}^1}(T[x], T[y]) = 2 \cdot d_{\mathbb{T}^1}([x], [y]).$$

(c) Show that a positively expansive continuous map on a compact metric space X has finitely many fixed points.

Q8 Let $n \in \mathbb{N}_{\geq 2}$. Recall that the **one-sided full-shift (on n symbols)** is the topological dynamical system $(\{1, 2, \ldots, n\}^{\mathbb{N}}, \sigma)$ where $\{1, 2, \ldots, n\}^{\mathbb{N}}$ is the collection of all one-sided infinite sequences $(x_k)_{k \in \mathbb{N}}$ with entries $x_k \in \{1, 2, \ldots, n\}$ and σ is the left shift.

(a) Show that the one-sided full-shift is positively expansive.

(b) Compute the topological entropy h of the one-sided full-shift on n symbols.

(c) Is there a subshift of $(\{1, 2, \ldots, n\}^{\mathbb{N}}, \sigma)$ whose topological entropy is bigger than the topological entropy h of the one-sided full-shift on n symbols? Justify your answer.