



EXAMINATION PAPER

Examination Session: May/June	Year: 2025	Exam Code: MATH43720-WE01
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Title: Stochastic Analysis V
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Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

Q1 Let $(W_t)_{t \geq 0}$ be a Brownian motion in \mathbb{R} .

(a) Justify your answers as to whether each of the following stochastic processes is a Brownian motion or not.

(i) $X_t := 2W_{t/4}$

(ii) $Y_t := W_{2t} - W_t$

(iii) $Z_t := \sqrt{t}W_1$

(b) Prove that the following stochastic process is a Brownian motion:

$$B_t := \begin{cases} tW_{1/t}, & t > 0 \\ 0, & t = 0 \end{cases}$$

Hint: Carefully discuss the continuity of $t \mapsto B_t$ at $t = 0$.

Q2 (a) Let $X = (X_n)_{n \in \mathbb{Z}_+}$ be a discrete-time martingale with respect to a filtration $(\mathcal{F}_n)_{n \in \mathbb{Z}_+}$, such that $X_0 = 0$. Show that

$$\mathbb{E}[X_n^2] = \sum_{k=0}^{n-1} \mathbb{E}[(X_{k+1} - X_k)^2]$$

for each $n \in \mathbb{N}$.

Now let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of independent random variables, such that

$$\mathbb{E}[Z_n] = 0 \quad \text{and} \quad \text{Var}(Z_n) = \gamma^n$$

for each $n \in \mathbb{N}$, for some $\gamma \in (0, 1)$, and let $\mathcal{F}_n = \sigma(Z_1, Z_2, \dots, Z_n)$, $n \in \mathbb{Z}_+$, be the natural filtration of this sequence. Let $M_0 = 0$, and

$$M_n = \sum_{k=1}^n Z_k$$

for each $n \in \mathbb{N}$.

(b) Show that $M = (M_n)_{n \in \mathbb{Z}_+}$ is a martingale with respect to $(\mathcal{F}_n)_{n \in \mathbb{Z}_+}$.

(c) Use the result of part (a) to show that M is bounded in L^2 .

Q3 In your answers to the following parts, you may use, without proof, the result that any bounded and continuous real-valued martingale M is of finite quadratic variation and $\langle M, M \rangle$ is the unique continuous, adapted and increasing real-valued process such that $M^2 - \langle M, M \rangle$ is a martingale.

(a) Let $M_t, t \in \mathbb{R}_+$, be a continuous real-valued local martingale. Prove that there is a unique continuous increasing adapted process $\langle M, M \rangle$, vanishing at zero, such that $M^2 - \langle M, M \rangle$ is a continuous local martingale.

(b) Show that if $M_t, t \in \mathbb{R}_+$, is a continuous and bounded real-valued martingale, then

$$\mathbb{E}[(M_t - M_0)^2] = \mathbb{E}[\langle M, M \rangle_t].$$

- Q4** (a) Let B be a Brownian motion on \mathbb{R} , and K be a progressive process on the same filtered probability space satisfying, for every $t > 0$,

$$\mathbb{E} \left[\int_0^t K_s^2 ds \right] < \infty.$$

Denote by H^2 the set of the L^2 -bounded and continuous martingales (i.e., all continuous martingales M satisfying $\sup_{t \in \mathbb{R}_+} \mathbb{E}[M_t^2] < \infty$), and H_0^2 the subset of H^2 vanishing at 0.

Use the stopping technique and stochastic integration theory on H_0^2 for a progressive process, say \tilde{K} , satisfying

$$\mathbb{E} \left[\int_0^\infty \tilde{K}_s^2 ds \right] < \infty,$$

to construct the stochastic integral $\int_0^t K_s dB_s$ for all $t > 0$.

- (b) Verify that you can use part (a) of this question to define $M_t = \int_0^t B_s dB_s$.
 (c) Calculate $\langle M, M \rangle_t$ and $\mathbb{E}[M_t^2]$.

SECTION B

- Q5** (a) State the definition that a continuous function $u: [0, T] \rightarrow \mathbb{R}$ has a finite quadratic variation.
 (b) Let $u: [0, T] \rightarrow \mathbb{R}$ be a C^1 -function. Prove that its quadratic variation is zero.
 (c) Let $(W_t)_{t \in [0, T]}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define

$$[W]_{\mathbf{t}, T} := \sum_{i=0}^{n_{\mathbf{t}}-1} |W_{t_{i+1}} - W_{t_i}|^2,$$

where $\mathbf{t} = (t_i)_{i=0}^{n_{\mathbf{t}}}$ with $0 = t_0 < t_1 < \dots < t_{n_{\mathbf{t}}} = T$ is any partition of $[0, T]$. Prove that

$$\lim_{|\mathbf{t}| \rightarrow 0} [W]_{\mathbf{t}, T} = T \quad \text{in } L^2(\Omega, \mathbb{P}).$$

Q6 Let $(\xi_i)_{i \in \mathbb{N}}$ be a sequence of independent and identically distributed random variables, such that

$$\mathbb{P}(\xi_i = 1) = p \quad \text{and} \quad \mathbb{P}(\xi_i = -1) = 1 - p$$

for some $p \in (0, 1)$, and let $\mathcal{F}_n = \sigma(\xi_1, \xi_2, \dots, \xi_n)$, $n \in \mathbb{Z}_+$, be the natural filtration of this sequence. Let $S_0 = 0$ and

$$S_n = \sum_{i=1}^n \xi_i$$

for each $n \in \mathbb{N}$. Let

$$T = \inf\{n \in \mathbb{Z}_+ : S_n = k\}$$

for some $k \in \mathbb{N}$, and let $X_n = 2^{S_n}$ for each $n \in \mathbb{Z}_+$.

- (a) Show that the process $X = (X_n)_{n \in \mathbb{Z}_+}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n \in \mathbb{Z}_+}$ for a particular value of p , which you should identify.
- (b) State Doob's supermartingale convergence theorem, and then use it to show, with the value of p found in part (a), that the limit

$$\lim_{n \rightarrow \infty} X_{n \wedge T}$$

exists almost surely.

- (c) Show that this limit is given by

$$\lim_{n \rightarrow \infty} X_{n \wedge T} = 2^k \mathbb{1}_{\{T < \infty\}}.$$

- (d) Using the result of part (c), prove that

$$\mathbb{P}\left(\sup_{n \in \mathbb{Z}_+} S_n \geq k\right) = 2^{-k}.$$

Q7 Let $X_t, t \geq 0$, be a real-valued submartingale such that $\mathbb{E}[|X_t|] < \infty$ for every $t \geq 0$. For some $t \in (0, \infty)$, let $t_n, n = 1, 2, \dots$, be a sequence decreasing to t , i.e., such that $t_n \searrow t$ as $n \rightarrow \infty$.

- (a) Use the submartingale downcrossing inequality to prove that

$$X_{t+}(\omega) := \lim_{n \rightarrow \infty} X_{t_n}(\omega)$$

exists for almost every $\omega \in \Omega$.

- (b) Prove that $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_{t_n}|] < \infty$.
- (c) Show that $X_{t_n}, n = 1, 2, \dots$, is uniformly integrable with respect to expectation.
- (d) Prove that $\mathbb{E}[|X_{t+}|] < \infty$.

Q8 Let $X_t^x, t \in \mathbb{R}_+$, be the solution of the following stochastic differential equation

$$dX_t^x = -X_t^x dt + dW_t, \quad X_0^x = x,$$

where $W_t, t \in \mathbb{R}_+$, is a standard Brownian motion on \mathbb{R} , and $x \in \mathbb{R}$. For any bounded measurable function $f: \mathbb{R} \rightarrow \mathbb{R}$, let

$$P_t f(x) = \mathbb{E}[f(X_t^x)], \quad t \in \mathbb{R}_+, x \in \mathbb{R}.$$

- (a) Prove, using the Markov property, that for any $t, s \geq 0$, and f being bounded and measurable,

$$P_t \circ P_s f = P_{t+s} f.$$

- (b) Verify that $X_t^x, t > 0$, is given by

$$X_t^x = \exp\{-t\}x + \int_0^t \exp\{-(t-s)\} dW_s.$$

And use this to show that for any $t > 0$, X_t^x is a Gaussian random variable with mean $e^{-t}x$ and variance $\frac{1}{2}(1 - e^{-2t})$, and

$$P_t f(x) = \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \int_{\mathbb{R}} f(y) \exp\left\{-\frac{(y - e^{-t}x)^2}{1 - e^{-2t}}\right\} dy.$$

- (c) Let A be the infinitesimal generator of $X_t^x, t \in \mathbb{R}_+$. Prove that

$$Af = \frac{1}{2} \frac{d^2}{dx^2} f - x \frac{d}{dx} f$$

when $f \in C^2$ is bounded and has bounded continuous derivatives up to second order.