EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH43920-WE01

Title:
Topics in Combinatorics V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 (a) Let $n \geq 2$ be an integer. Compute the number of Dyck paths of length $2n$ whose first peak is at height 2.

(b) Denote by $p_k(n)$ the number of Young diagrams $\lambda \vdash n$ with exactly k rows. Show that

$$p_k(n) = p_k(n-k) + p_{k-1}(n-k) + p_{k-2}(n-2)$$

for all $n > k$ and $k > 2$.

Hint: Partition the $p_k(n)$ diagrams according to the number of boxes in rows $k-1$ and k.

Q2 (a) Let $w = 4\,6\,8\,5\,9\,3\,7\,1\,2 \in S_9$. Apply the Robinson–Schensted–Knuth (RSK) algorithm to compute the insertion and recording tableaux P and Q.

(b) Let (P', Q') be a pair of standard Young tableaux of shape $\lambda = (4, 2, 1, 1, 1) \vdash 9$, where

$$P' = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 9 \\ \hline 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1} 6 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \qquad Q' = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 7 & 8 \\ \hline 3 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} 6 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array}$$

(i) Find $w' \in S_9$ which is taken to the pair (P', Q') by the RSK algorithm.

(ii) Find $w'' \in S_9$ which is taken to the pair (Q', P') by the RSK algorithm.

Q3 Let Δ be a root system of type A_3.

(a) Find the Coxeter number of Δ.

(b) Let W be the Weyl group of Δ, i.e. $W = \langle s_1, s_2, s_3 \mid s_i^2, (s_1s_2)^3, (s_2s_3)^3, (s_1s_3)^2 \rangle$. Show that the subgroup of W generated by s_2 and s_1s_3 is dihedral and find its order.

Q4 Let Δ be a root system of type B_4, and denote by Δ_l the set of long roots of Δ.

(a) Show that Δ_l is a root system and find its type.

(b) Let W be the Weyl group of Δ_l. Show that any two reflections in W are conjugated to each other, i.e., for any two reflections r_1 and r_2 there exists $g \in W$ such that $r_1 = gr_2g^{-1}$.

SECTION B

Q5 Given a permutation $w = w_1 \cdots w_n \in S_n$, recall that an *inversion* of w is a pair of integers $i, j \in \{1, \ldots, n\}$ with $i < j$ and $w_i > w_j$. Let $\mathrm{inv}(w)$ denote the number of inversions of w.

(a) Calculate $\mathrm{inv}(w)$ for each permutation $w \in S_3$, and for $n = 3$ show that the generating function $f_{\mathrm{inv}}(x) := \sum_{w \in S_n} x^{\mathrm{inv}(w)}$ is of the form

$$f_{\mathrm{inv}}(x) = \prod_{k=0}^{2} g_k(x),$$

where g_k is a polynomial of degree k that you should determine.

(b) State and prove a formula for $f_{\mathrm{inv}}(x)$ that holds for general $n \in \mathbb{N}$.

(c) Define the *Major index* $\mathrm{maj}(w)$ of a permutation $w \in S_n$, and show that maj and inv are equidistributed permutation statistics.

Q6 (a) Carefully define the Young lattice $\mathbb{Y}$, and prove that it is indeed a lattice. Determine whether $\mathbb{Y}$ is a distributive lattice.

(b) Fix the Young diagram $\lambda = (3, 1) \vdash 4$, and let P_λ be the poset on boxes (i, j) of λ with $(i, j) \leq_{P_\lambda} (k, l)$ if and only if $i \leq k$ and $j \leq l$. Draw the Hasse diagram of P_λ and its poset of order ideals $J(P_\lambda)$.

(c) Prove that for any Young diagram μ the number of saturated chains in $\mathbb{Y}$ from $\emptyset$ to μ equals the number of standard Young tableaux of shape μ.

Q7 Let (G, S) be a Coxeter system, $G = \langle S \mid s_i^2, (s_i s_j)^{m_{ij}} \rangle$, and let $T \subset S$. Define G_T to be the subgroup of G generated by elements of T. (We call G_T a *standard parabolic subgroup* of G.)

(a) Let $w = s_1 \ldots s_k$ be a word, with all $s_i \in T$. Show that for any M-reduction $w \to w_0$ all words obtained during the procedure belong to G_T.

(b) Let $\Gamma = \langle T \mid s_i^2, (s_i s_j)^{m_{ij}} \rangle$. Define a homomorphism $\phi : \Gamma \to G$ by $\phi(s_i) = s_i$. Show that $\ker \phi$ is trivial.

(c) Show that (G_T, T) is a Coxeter system.

Q8 Let Δ be a root system of type C_3.

(a) Compute the Coxeter number of Δ and the exponents of the Weyl group of Δ.

(b) Let P be the root poset of Δ. Draw the Hasse diagram of P.

(c) Draw the Hasse diagram of the poset of order ideals of P. Identify join-irreducible elements.

