Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH4411-WE01

Title: Advanced Mathematical Biology IV

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Consider the process $X = \{X(t) \in (0,\infty) \,:\, t \geq 0\}$ obeying the following stochastic differential equation (written in terms of its computational definition):

$$X(t+\Delta t) = X(t) + (\alpha X(t) - \beta X(t)^3)\Delta t + \sigma\sqrt{X(t)}\sqrt{\Delta t}\,\xi, \quad t > 0, \tag{1}$$

where α, β, and σ are positive parameters, $\xi \sim N(0,1)$ and the process is subject to a reflecting boundary at $x = 0$.

(a) Write pseudocode to simulate paths of the SDE (1).

(b) Write down the corresponding Fokker-Planck equation for (1) along with any necessary boundary conditions.

(c) Calculate the stationary distribution of the process X obeying (1) up to a constant factor. State any normalisation constants explicitly.

Q2 A single chemical species A reacts in a container of volume ν with the following reaction dynamics:

$$2A \xrightarrow{k_1} A, \quad A \xrightarrow{k_2} \emptyset, \quad \emptyset \xrightarrow{k_3} A. \tag{2}$$

(a) Write the corresponding ordinary differential equation (ODE) for the concentration $a(t)$ of A using the law of mass action.

(b) Suppose $k_1/\nu = 0.02\,\mathrm{min}^{-1}$, $k_2 = 0.1\,\mathrm{min}^{-1}$, and $k_3\nu = 5\,\mathrm{min}^{-1}$. Analyze the steady-state behavior of the mass-action system and determine whether the stationary distribution of the stochastic reaction process is likely to be unimodal or multimodal. Explain your reasoning.

(c) Suppose that the molecules are confined to a one-dimensional spatial domain $\Omega = [0, L]$ and diffuse in space at a diffusion rate D.

Defining any notation that you introduce, explain how you could model this process using a spatial discretisation and an appropriate chemical reaction process.

Q3 Consider pulsatile flow of blood in an artery, modelled as a (horizontal) cylindrical tube of radius R and length L. A cylindrical coordinate system (r, θ, x) where $r \in [0, R]$, $\theta \in [0, 2\pi]$, and $x \in [0, L]$ is considered. The following assumptions are made. The blood is modelled as an incompressible and Newtonian fluid. The flow is considered to be laminar and axisymmetric, and it is driven by a periodic pressure gradient along the x-direction. Entrance effects are negligible. The effect of gravity on the fluid flow is negligible.

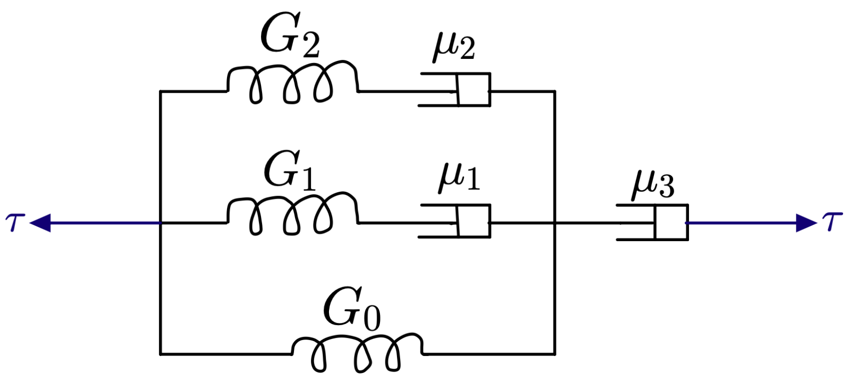
(a) Give the definition of the following terms: incompressible flow, laminar flow, and axisymmetric flow. What does the assumption of negligible entrance effects physically mean?

(b) Show that the x-component of the Navier-Stokes equation reduces to a Bessel-like ordinary differential equation of the form

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} + \frac{1}{X}\frac{\mathrm{d}Y}{\mathrm{d}X} + c_1 Y + c_2 = 0,$$

where $c_1,\ c_2$ are constants.
HINT: The periodic pressure gradient can be represented by Fourier series.

Q4 Bones can be modelled as viscoelastic materials whose rheological behaviour is described by the mechanical analog model shown below.



(a) State the viscoelastic elements composing the model shown above, and their connection.

(b) Show that the constitutive equation of the mechanical analog model takes the form

$$C_{\tau_1}\dddot{\tau} + C_{\tau_2}\ddot{\tau} + C_{\tau_3}\dot{\tau} + C_{\tau_4}\tau = C_{\gamma_1}\dddot{\gamma} + C_{\gamma_2}\ddot{\gamma} + C_{\gamma_3}\dot{\gamma},$$

with the constants $C_{\tau_{1-4}}$ and $C_{\gamma_{1-3}}$ given by

$$C_{\tau_1} = \frac{\mu_3}{G_1 G_2},\ C_{\tau_2} = \frac{\mu_3}{\mu_1 G_2} + \frac{\mu_3}{\mu_2 G_1} + \frac{G_0}{G_1 G_2} + \frac{1}{G_1} + \frac{1}{G_2},\ C_{\tau_3} = \frac{\mu_3}{\mu_1 \mu_2} + \frac{G_0}{\mu_1 G_2} + \frac{G_0}{\mu_2 G_1} + \frac{1}{\mu_1} + \frac{1}{\mu_2},\ C_{\tau_4} = \frac{G_0}{\mu_1 \mu_2},$$

$$C_{\gamma_1} = \frac{\mu_3 G_0}{G_1 G_2} + \frac{\mu_3}{G_1} + \frac{\mu_3}{G_2},\ C_{\gamma_2} = \frac{\mu_3 G_0}{\mu_1 G_2} + \frac{\mu_3 G_0}{\mu_2 G_1} + \frac{\mu_3}{\mu_2} + \frac{\mu_3}{\mu_1},\ C_{\gamma_3} = \frac{\mu_3 G_0}{\mu_1 \mu_2}.$$

The parameters μ_1, μ_2, and μ_3 denote viscosities, G_0, G_1, and G_2 denote elastic moduli, τ and $\dot{\gamma}$ represent the stress and strain, respectively.

SECTION B

Q5 Consider a two-species chemical system within a well-mixed container of volume ν with the following reactions:

$$\emptyset \xrightarrow{k_1} \text{A}, \quad \text{A} \xrightarrow{k_2} \text{B}, \quad \text{B} \xrightarrow{k_3} \emptyset$$

Let $A(t)$ and $B(t)$ denote the number of molecules of species A and B, respectively, at time t. Assume $A(0) = 0$ and $B(0) = 0$. Define $P_{n,m}(t) = \mathbb{P}[A(t) = n, B(t) = m]$ for $n, m \geq 0$.

(a) Write down the chemical master equations for this two-species process.

(b) Define the joint probability generating function (PGF) for this process as:

$$G(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x^n y^m P_{n,m}(t)$$

Derive an evolution equation for $G(x, y, t)$ from the chemical master equations.

(c) Find the steady-state joint PGF, $G_s(x, y)$ using a separation of variables approach by showing that the marginal steady-state PGFs for species A and B are Poisson.

HINT: The PGF of a Poisson random variable with rate parameter $\lambda > 0$ is given by $G(z) = \exp(\lambda(z - 1))$ for $z \in (-1, 1)$.

(d) Are the number of molecules of A and B independent? Justify your answer.

(e) Now assume there is an additional reaction where species B can be converted back to species A:

$$\text{B} \xrightarrow{k_4} \text{A}$$

Do you expect the stationary marginal distributions to remain Poisson in this case? Explain your reasoning without performing further calculations.

Q6 Consider a chemical system within a well-mixed container of volume ν involving two species, X and Y. The reaction dynamics are given by:

$$\emptyset \xrightarrow{k_1} X, \quad X \xrightarrow{k_2} \emptyset, \quad X + Y \xrightarrow{k_3} 2Y, \quad Y \xrightarrow{k_4} \emptyset$$

Suppose that the reaction rates of the system are given by:

$$k_1\nu = 10 \text{ min}^{-1}, \quad k_2 = 0.2 \text{ min}^{-1}, \quad k_3/\nu = 0.01 \text{ min}^{-1}, \quad k_4 = 0.5 \text{ min}^{-1}.$$

(a) Explain how you could interpret this process as a stochastic model of predator-prey interactions.

(b) Write down a deterministic model for this process based on the law of mass action and determine the steady-state concentrations of X and Y.

(c) Assume that the concentration of Y is held constant at its steady-state value obtained in part (b), denoted as y_{ss}. Write down a continuum approximation of this reduced reaction process in terms of the appropriate chemical Fokker-Planck equation and state when this approximation is valid.

(d) Suppose the system starts with zero X molecules. Using the chemical Fokker-Planck equation derived in part (c), estimate the mean time until the concentration of X reaches a level of 50 molecules per unit volume.

HINT: The hitting time cannot be evaluated in closed form but can be expressed in the form

$$\int_a^b \phi(z) \int_{l(z)}^{u(z)} \psi(x)\, dx\, dz,$$

for some elementary functions ϕ and ψ.

Q7 Consider blood flowing in a counter-rotating concentric cylinder rheometer. The outer cylinder of radius R_2 is rotated at a constant angular velocity Ω, while the inner cylinder of radius R_1 is rotated in the opposite direction with the same angular velocity. Assume that: blood is modelled as an incompressible, Newtonian fluid, the blood flow is steady, laminar and axisymmetric, end effects are negligible, no pressure gradients arise, and the effect of gravity on the fluid flow is negligible.

(a) Show that the azimuthal velocity u_θ of blood in the counter-rotating concentric cylinder rheometer is given by the following profile:

$$u_\theta\left(r\right) = \frac{\Omega}{R_2^2 - R_1^2}\left[\left(R_1^2 + R_2^2\right)r - \frac{2R_1^2R_2^2}{r}\right].$$

(b) In the special case of a narrow gap between the two concentric cylinders, find the simplified expression for the azimuthal velocity u_θ profile and the shear stress $\tau_{r\theta}$ distribution.
HINT: The same simplifications for u_θ and $\tau_{r\theta}$ can be considered here, as those seen in the simple concentric cylinder rheometer.

(c) In the above analysis, we have assumed that the blood under examination behaves like a Newtonian fluid. Considering the special case of a narrow gap between the two concentric cylinders of radius $R_1 = 10.0$ mm and $R_2 = 10.1$ mm, rotating at a constant angular velocity $\Omega = 10.0$ rad/s, and given that blood behaves like a Newtonian fluid for shear strain rates < 100 s^{-1}, is this assumption correct? Justify your answer. If the assumption is not correct, which non-Newtonian fluid models would be better to consider?

Q8 Consider blood flowing in the descending aorta. The descending aorta is modelled as a vertical, rigid cylindrical tube of radius R and length L. A cylindrical coordinate system (r, θ, x) where $r \in [0, R]$, $\theta \in [0, 2\pi]$, and $x \in [0, L]$ is considered. The following assumptions are made: blood is an incompressible fluid, the flow is steady, laminar and axisymmetric, no pressure gradients arise, and the flow is driven by gravity g acting only on the positive x-direction.

(a) The blood is modelled as a power-law fluid. State the constitutive equation of a power-law fluid, defining all the parameters that appear in it and explaining their physical meaning.

(b) Show that the shear stress τ_{rx} distribution is given by

$$\tau_{rx}\left(r\right) = -\frac{\rho g r}{2},$$

where ρ is the blood density.

(c) Derive the axial velocity $u_x\left(r\right)$ profile of blood flowing in the descending aorta.