

Formula Sheet for MATH 4421

Geophysical and Astrophysical Fluids IV

1 Some vector identities

$$\nabla \cdot (f\mathbf{A}) = (\nabla f) \cdot \mathbf{A} + f\nabla \cdot \mathbf{A} \quad (\text{F1})$$

$$\nabla \times (f\mathbf{A}) = (\nabla f) \times \mathbf{A} + f\nabla \times \mathbf{A} \quad (\text{F2})$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (\text{F3})$$

$$(\mathbf{A} \cdot \nabla) \mathbf{A} = \frac{1}{2} \nabla |\mathbf{A}|^2 - \mathbf{A} \times (\nabla \times \mathbf{A}) \quad (\text{F4})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \quad (\text{F5})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} \quad (\text{F6})$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \quad (\text{F7})$$

2 In cylindrical coordinates (r, θ, z)

$$\nabla f = \mathbf{e}_r \partial_r f + \frac{\mathbf{e}_\theta}{r} \partial_\theta f + \mathbf{e}_z \partial_z f \quad (\text{F8})$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \partial_r (r A_r) + \frac{1}{r} \partial_\theta A_\theta + \partial_z A_z \quad (\text{F9})$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r} \partial_\theta A_z - \partial_z A_\theta \right) \mathbf{e}_r + (\partial_z A_r - \partial_r A_z) \mathbf{e}_\theta + \frac{1}{r} (\partial_r (r A_\theta) - \partial_\theta A_r) \mathbf{e}_z \quad (\text{F10})$$

$$\nabla^2 f = \frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r^2} \partial_{\theta\theta} f + \partial_{zz} f \quad (\text{F11})$$

$$\nabla^2 \mathbf{A} = \left(\nabla^2 A_r - \frac{1}{r^2} A_r - \frac{2}{r^2} \partial_\theta A_\theta \right) \mathbf{e}_r + \left(\nabla^2 A_\theta + \frac{2}{r^2} \partial_\theta A_r - \frac{1}{r^2} A_\theta \right) \mathbf{e}_\theta + \mathbf{e}_z \nabla^2 A_z \quad (\text{F12})$$

$$(\mathbf{B} \cdot \nabla) \mathbf{A} = \mathbf{e}_r \left(\mathbf{B} \cdot \nabla A_r - \frac{B_\theta A_\theta}{r} \right) + \mathbf{e}_\theta \left(\mathbf{B} \cdot \nabla A_\theta + \frac{B_\theta A_r}{r} \right) + \mathbf{e}_z \mathbf{B} \cdot \nabla A_z \quad (\text{F13})$$

3 In spherical coordinates (r, θ, ϕ)

$$\nabla f = \mathbf{e}_r \partial_r f + \frac{\mathbf{e}_\theta}{r} \partial_\theta f + \frac{\mathbf{e}_\phi}{r \sin \theta} \partial_\phi f \quad (\text{F14})$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \partial_r (r^2 A_r) + \frac{1}{r \sin \theta} \partial_\theta (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \partial_\phi A_\phi \quad (\text{F15})$$

$$\nabla \times \mathbf{A} = \frac{\mathbf{e}_r}{r \sin \theta} (\partial_\theta (A_\phi \sin \theta) - \partial_\phi A_\theta) + \frac{\mathbf{e}_\theta}{r} \left(\frac{1}{\sin \theta} \partial_\phi A_r - \partial_r (r A_\phi) \right) + \frac{\mathbf{e}_\phi}{r} (\partial_r (r A_\theta) - \partial_\theta A_r) \quad (\text{F16})$$

$$\nabla^2 f = \frac{1}{r} \partial_{rr} (r f) + \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta f) + \frac{1}{r^2 \sin^2 \theta} \partial_{\phi\phi} f \quad (\text{F17})$$

$$\begin{aligned} \nabla^2 \mathbf{A} = & \left(\nabla^2 A_r - \frac{2}{r^2} A_r - \frac{2}{r^2 \sin \theta} [\partial_\theta (\sin \theta A_\theta) + \partial_\phi A_\phi] \right) \mathbf{e}_r \\ & + \left(\nabla^2 A_\theta + \frac{2}{r^2} \partial_\theta A_r - \frac{A_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \partial_\phi A_\phi \right) \mathbf{e}_\theta \\ & + \left(\nabla^2 A_\phi + \frac{2}{r^2 \sin \theta} \partial_\phi A_r + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \partial_\phi A_\theta - \frac{A_\phi}{r^2 \sin^2 \theta} \right) \mathbf{e}_\phi \end{aligned} \quad (\text{F18})$$

$$\begin{aligned} (\mathbf{B} \cdot \nabla) \mathbf{A} = & \mathbf{e}_r \left(\mathbf{B} \cdot \nabla A_r - \frac{B_\theta A_\theta}{r} - \frac{B_\phi A_\phi}{r} \right) + \mathbf{e}_\theta \left(\mathbf{B} \cdot \nabla A_\theta - \frac{B_\phi A_\phi}{r} \cot \theta + \frac{B_\theta A_r}{r} \right) \\ & + \mathbf{e}_\phi \left(\mathbf{B} \cdot \nabla A_\phi + \frac{B_\phi A_r}{r} + \frac{B_\theta A_\theta}{r} \cot \theta \right) \end{aligned} \quad (\text{F19})$$

4 Bessel functions

Bessel functions $u(r) = J_n(r)$ and $u(r) = Y_n(r)$ are solutions to the ODE

$$r^2 u'' + r u' + (r^2 - n^2) u = 0. \quad (\text{F20})$$

Both $J_n(r)$ and $Y_n(r) \rightarrow 0$ as $r \rightarrow \infty$; $J_n(0) = \delta_{n0}$, and $|Y_n(r)| \rightarrow \infty$ as $r \rightarrow 0$.

5 Legendre polynomials

When ℓ is an integer, the ODE

$$(1 - s^2) u'' - 2s u' + \ell(\ell + 1) u = 0, \quad (\text{F21})$$

has polynomial solutions $P_\ell(s)$.

[The other solutions $Q_\ell(s)$ are singular at $s = \pm 1$, and for non-integer ℓ neither set of solutions is well defined at $s = \pm 1$.]

6 Transport theorems

$$\frac{d}{dt} \int_{V_t} f \, dV = \int_{V_t} \left(\frac{Df}{Dt} + f \nabla \cdot \mathbf{u} \right) dV \quad [\text{for a material volume } V_t] \quad (\text{F22})$$

$$\frac{d}{dt} \int_{S_t} \mathbf{b} \cdot d\mathbf{S} = \int_{S_t} \left(\frac{\partial \mathbf{b}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{b}) + \mathbf{u} \nabla \cdot \mathbf{b} \right) \cdot d\mathbf{S} \quad [\text{for a material surface } S_t] \quad (\text{F23})$$

$$\frac{d}{dt} \int_{\gamma_t} \mathbf{a} \cdot d\mathbf{l} = \int_{\gamma_t} \left(\frac{\partial \mathbf{a}}{\partial t} - \mathbf{u} \times \nabla \times \mathbf{a} + \nabla(\mathbf{u} \cdot \mathbf{a}) \right) \cdot d\mathbf{l} \quad [\text{for a material curve } \gamma_t] \quad (\text{F24})$$