Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH4421-WE01

Title: Geophysical and Astrophysical Fluids IV

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Consider an inertial frame defined by the orthogonal basis $\boldsymbol{e}_i = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and a rotating frame with the orthogonal basis $\hat{\boldsymbol{e}}_i = \{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$. The axis of rotation for the rotating frame is aligned with $\hat{\boldsymbol{e}}_2 = \sqrt{2}/2\boldsymbol{e}_2 + \sqrt{2}/2\boldsymbol{e}_3$ and rotates with a constant angular velocity of $\Omega = 1$. The vectors $\boldsymbol{e}_1$ and $\hat{\boldsymbol{e}}_1$ are initially aligned (only at $t = 0$).

(a) Find an expression for $\left(\frac{d}{dt}(2\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 - 4\hat{\boldsymbol{e}}_3)\right)_{\mathrm{I}}$ (where subscript "I" denotes the inertial frame). Express your final answer in terms of $\hat{\boldsymbol{e}}_i$. Recall the relation between the time derivatives in the rotating and inertial frame

$$\left(\frac{d\mathbf{X}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\mathbf{X}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \mathbf{X}.$$

(b) It can be shown that

$$\begin{aligned}
\boldsymbol{e}_1 &= \cos(t)\hat{\boldsymbol{e}}_1 + \sin(t)\hat{\boldsymbol{e}}_3, \\
\boldsymbol{e}_2 &= \frac{\sqrt{2}}{2}\sin(t)\hat{\boldsymbol{e}}_1 + \frac{\sqrt{2}}{2}\hat{\boldsymbol{e}}_2 - \frac{\sqrt{2}}{2}\cos(t)\hat{\boldsymbol{e}}_3, \\
\boldsymbol{e}_3 &= \frac{\sqrt{2}}{2}\cos(t)\hat{\boldsymbol{e}}_3 - \frac{\sqrt{2}}{2}\sin(t)\hat{\boldsymbol{e}}_1 + \frac{\sqrt{2}}{2}\hat{\boldsymbol{e}}_2.
\end{aligned}$$

If a particle's position vector is defined as $\boldsymbol{X} = 10\,t\,\boldsymbol{e}_2$ in the inertial frame, find its acceleration in the rotating frame.

Q2 Consider a two-dimensional flow (independent of z) that is in geostrophic balance,

$$f_0 \boldsymbol{e}_z \times \boldsymbol{v} = -\frac{1}{\rho_0} \nabla_{\mathrm{H}}\, p,$$

where $\boldsymbol{v} = (u, v)$ is the horizontal velocity and f_0 the constant Coriolis parameter. Because this flow is divergence-free, we can define a streamfunction ψ such that

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}.$$

(a) If $\hat{\boldsymbol{v}}(\boldsymbol{k})$ and $\hat{\psi}(\boldsymbol{k})$ are the Fourier transforms of $\boldsymbol{v}$ and ψ respectively, find an expression for $\hat{\boldsymbol{v}}(\boldsymbol{k})\hat{\boldsymbol{v}}(\boldsymbol{k})^*$ in terms of $\hat{\psi}(\boldsymbol{k})$. Note that * denotes the complex conjugate, and $\boldsymbol{k} = (k_x, k_y)$ is the 2D wavevector in Fourier space. (Remember that the Fourier transform of the function f is defined as $\hat{f}(\boldsymbol{k}) = \int_{-\infty}^{\infty} f(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, d\boldsymbol{x}$).

(b) Considering that $\zeta = \partial v/\partial x - \partial u/\partial y$ is the vertical vorticity, derive an expression for $\hat{\zeta}(\boldsymbol{k})\hat{\zeta}(\boldsymbol{k})^*$ in terms of $\hat{\psi}(\boldsymbol{k})$.

(c) Prove that

$$Z(k_h) = k_h^2 E(k_h),$$

where

$$E(k_h) = \frac{1}{2}\int_{\Gamma(k_h)} \hat{\boldsymbol{v}}(\boldsymbol{k})\hat{\boldsymbol{v}}(\boldsymbol{k})^* d\boldsymbol{s}, \quad Z(k_h) = \frac{1}{2}\int_{\Gamma(k_h)} \hat{\zeta}(\boldsymbol{k})\hat{\zeta}(\boldsymbol{k})^* d\boldsymbol{s}.$$

In the above equations, $k_h = \sqrt{k_x^2 + k_y^2}$, and the integration is carried over the circles with the radius k_h denoted by $\Gamma(k_h)$. You should clarify all the mathematical steps you take in this part.

Q3 (a) Show that for magnetic fields of the form $\boldsymbol{B}(x, y) = \nabla \times (A(x, y)\boldsymbol{e}_z)$, magnetic field lines are curves of constant A.

(b) For $\boldsymbol{B}(x, y) = \dfrac{2y}{b^2}\boldsymbol{e}_x - \dfrac{2x}{a^2}\boldsymbol{e}_y$, where a and b are positive real numbers, find $A(x, y)$.

(c) Sketch the magnetic field lines for $a > b$, labelling all axis crossings and indicating the direction of the field.

Q4 (a) Explain what is meant by a perfectly conducting fluid and show that, for such a fluid, which is also incompressible, the induction equation becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}.$$

(b) If $\boldsymbol{u} = x\boldsymbol{e}_x - y\boldsymbol{e}_y$ and $\boldsymbol{B}(x, y, 0) = y^2\boldsymbol{e}_x$ at $t = 0$, show that under a perfectly conducting time evolution, $\boldsymbol{B}$ remains of the form $\boldsymbol{B} = B_x(y, t)\boldsymbol{e}_x$ and find $\boldsymbol{B}(x, y, t)$ for $t > 0$.

SECTION B

Q5 The two-dimensional viscous non-rotating non-hydrostatic Boussinesq equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{1a}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right), \tag{1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1c}$$

$$\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial x} + w\frac{\partial b}{\partial z} + N^2 w = \nu\left(\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial z^2}\right), \tag{1d}$$

(which are derived by setting $\partial/\partial y = 0$ and $v = 0$). We assume ρ_0 and ν are constant. To non-dimensionalise these equations, we consider scalings

$$x = x^*\mathcal{L}, \quad z = z^*\mathcal{H}, \quad t = t^*\mathcal{T},$$
$$u = u^*\,\mathcal{U}, \quad w = w^*(\mathcal{U}\mathcal{H}/\mathcal{L}), \quad p = p^*(\rho_0\mathcal{U}^2), \quad b = b^*\mathcal{B} \tag{2}$$

where * variables are dimensionless.

(a) Find the appropriate timescale $\mathcal{T}$ such that the time derivative of velocity $\dfrac{\partial u}{\partial t}$ and the advective term $u\dfrac{\partial u}{\partial x}$ have similar orders.

(b) Find $\mathcal{B}$ such that b and $\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial z}$ have similar orders.

(c) Using the scalings in (2) and the values of $\mathcal{T}$ and $\mathcal{B}$ that you found in parts (a) and (b), non-dimensionalise the equations (1). Your final answer should be in terms of the following dimensionless parameters

$$\frac{\mathcal{H}}{\mathcal{L}}, \quad \mathrm{Re} = \frac{\mathcal{U}\mathcal{L}}{\nu}, \quad \mathrm{Fr} = \frac{\mathcal{U}}{N\mathcal{H}}.$$

Note that no other scaling value or dimensionless parameter should remain in your equations.

(d) Assume

$$\frac{\mathcal{H}}{\mathcal{L}} = \mathcal{O}(\epsilon), \quad \mathrm{Re} = \mathcal{O}(\epsilon^{-3}), \quad \mathrm{Fr} = \mathcal{O}(1)$$

with $\mathcal{O}$ showing the order of magnitude. For small $\epsilon \ll 1$, find the leading order equations (for all four equations in (1)). Then, rewrite the leading-order equations in the dimensional form.

Q6 Consider the (non-rotating) Shallow Water equations with flat bottom

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla_{\mathrm{H}})\boldsymbol{v} = -g\ \nabla_{\mathrm{H}}\eta, \tag{3a}$$

$$\frac{\partial \eta}{\partial t} + H\ \nabla_{\mathrm{H}} \cdot \boldsymbol{v} + \nabla_{\mathrm{H}} \cdot (\eta \boldsymbol{v}) = 0, \tag{3b}$$

where we set $z = 0$ at the mean surface level and H denotes the average water height. We consider the dynamics to consist of a background flow and waves

$$u = U + \epsilon u', \quad v = V + \epsilon v', \quad \eta = \epsilon \eta', \tag{4}$$

where $(U, V) = (\mathrm{e}^{-\epsilon^2 y^2}, \sin(\epsilon^2 x))$ is the velocity of the background flow and u', v' and η' are the velocity and height variation of the waves. We assume ϵ to be a small parameter meaning that the waves terms are smaller than the background flow, and the background flow slowly varies with x and y (for example $dU/dy = \mathcal{O}(\epsilon^2)$).

(a) Substitute (4) into (3) and linearise for the wave terms by neglecting the terms that are $\mathcal{O}(\epsilon^2)$ or $\mathcal{O}(\epsilon^3)$ (in other words, keep the leading order terms).

(b) Find the dispersion relation for the linearised equations (that you derive in part (a)) by assuming the following wave ansatz

$$u' = \tilde{u}\ e^{i(k_x x + k_y y - \omega t)}, \quad v' = \tilde{v}\ e^{i(k_x x + k_y y - \omega t)}, \quad \eta' = \tilde{\eta}\ e^{i(k_x x + k_y y - \omega t)}.$$

(c) Find the group velocity for each set of waves that you find in part (b).

Q7 Consider a force-free magnetic configuration in which the current density $\boldsymbol{J}$ is zero.

(a) Show that in this case the magnetic field can be written as $\boldsymbol{B} = \nabla \Phi$ for some scalar potential Φ which satisfies Laplace's equation $\nabla^2 \Phi = 0$.

(b) Consider a region V in spherical polar coordinates (r, θ, ϕ) given by $0 < r_0 < r$. If $\Phi(r, \theta)$ is a Legendre series of the form

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} R_l(r) P_l(\cos\theta),$$

find the general form of $R_l(r)$ such that $\nabla^2 \Phi = 0$.

(c) Find $\boldsymbol{B}$ in V satisfying $B_r(r_0, \theta, \phi) = (5\cos^3\theta - 3\cos\theta)\boldsymbol{e}_r$ and $|\boldsymbol{B}| \to 0$ as $r \to \infty$. *Hint: $5s^3 - 3s$ is a solution of the Legendre ODE for suitable l.*

Q8 Consider a layer of fluid lying between two horizontal planes at $z = 0$ and $z = 1$. In such a layer, the non-dimensional linearised equations governing perturbations to a basic state in incompressible rotating convection can be written as

$$\left(\frac{\partial}{\partial t} - \mathrm{Pr}\nabla^2\right)\nabla^2 w' = \mathrm{Ra}\mathrm{Pr}\nabla_H^2\theta - \mathrm{Ta}^{\frac{1}{2}}\mathrm{Pr}\frac{\partial \omega_z'}{\partial z},$$

$$\left(\frac{\partial}{\partial t} - \mathrm{Pr}\nabla^2\right)\omega_z' = \mathrm{Ta}^{\frac{1}{2}}\mathrm{Pr}\frac{\partial w'}{\partial z},$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = w',$$

where ω_z' is the z-component of the perturbation vorticity $\boldsymbol{\omega'} = \nabla \times \boldsymbol{u'}$ and w' and θ are the perturbations to the vertical velocity and temperature, respectively. Ra, Pr and Ta are the non-dimensional parameters governing the system.

(a) Assume that the boundaries are impermeable, free-slip, and held at fixed temperature. By seeking normal mode solutions of the form

$$w' = W_0 \sin(n\pi z) f(x,y) e^{st},$$
$$\omega_z' = Z_0 \cos(n\pi z) f(x,y) e^{st},$$
$$\theta = \Theta_0 \sin(n\pi z) f(x,y) e^{st},$$

where $f(x, y)$ satisfies $\nabla_H^2 f = -k_h^2 f$, show that the dispersion relation for s can be written as $s^3 + Bs^2 + Cs + D = 0$. You should determine B and C and show that

$$D = \mathrm{Pr}^2\left[(k_h^2 + n^2\pi^2)^3 - k_h^2\mathrm{Ra} + n^2\pi^2\mathrm{Ta}\right].$$

(b) Show that Ra at the onset of direct (non-oscillatory) convection is given by

$$\mathrm{Ra} = \frac{(k_h^2 + n^2\pi^2)^3 + n^2\pi^2\mathrm{Ta}}{k_h^2}.$$

(c) Show that the critical horizontal wavenumber, k_{h_c}, satisfies

$$2k_{h_c}^6 + 3\pi^2 k_{h_c}^4 = \pi^6 + \pi^2\mathrm{Ta}.$$

In the asymptotic limit of $\mathrm{Ta} \to \infty$, what is the dependence of k_{h_c} and the critical Rayleigh number on Ta?