EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH4431-WE01

Title:
Advanced Probability IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 For fixed constant $\lambda > 0$, let X_1 and X_2 be independent $\mathsf{Exp}(\lambda)$ random variables, and let $X_{(1)}$ and $X_{(2)}$ be the corresponding order statistics.

a) Show that $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent and find their distributions.

b) Compute $\mathsf{E}(X_{(2)} \mid X_{(1)} = x_1)$ and $\mathsf{E}(X_{(1)} \mid X_{(2)} = x_2)$.

c) Compute $\mathsf{E}(X_{(1)})$, $\mathsf{Var}(X_{(1)})$, $\mathsf{E}(X_{(2)})$, and $\mathsf{Var}(X_{(2)})$.

Q2 Let r balls be placed uniformly and independently into n boxes. Denote by X_i the number of balls in the ith box and by N the number of empty boxes.

a) Show that $\mathsf{E}(n^{-1}N) = \left(1 - \frac{1}{n}\right)^r$ and $\mathsf{Var}\left(n^{-1}N\right) \to 0$ as $n \to \infty$.

b) Find the fraction of the empty boxes in the limit when $r/n \to c > 0$ as $n \to \infty$.

c) Show that $\mathsf{P}(X_1 = k) = \binom{r}{k}(n-1)^{r-k}/n^r$ and identify the limit, as $n \to \infty$, of this probability under the assumption of part b).

d) Find the probability $\mathsf{P}(X_1 = k_1, X_2 = k_2)$; what happens in the limit $n \to \infty$ under the assumption of part b)?

Q3 a) Let $X \geq 0$ be a non-degenerate integer-valued random variable with finite second moment, $0 < \mathsf{E}(X^2) < \infty$. Show that

$$\mathsf{P}(X = 0) \leq \frac{\mathsf{Var}(X)}{(\mathsf{E}X)^2}.$$

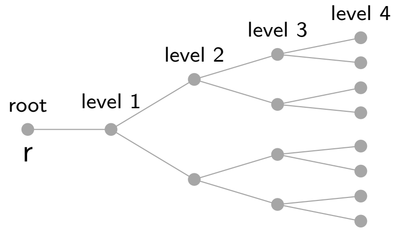
b) Let X and Y be random variables. Carefully show that $\big(\mathsf{E}(XY)\big)^2 \leq \mathsf{E}(X^2)\,\mathsf{E}(Y^2)$.

c) Let $X \geq 0$ be a non-degenerate integer-valued random variable with finite second moment, $0 < \mathsf{E}(X^2) < \infty$. Show that $\mathsf{P}(X > 0)\,\mathsf{E}(X^2) \geq (\mathsf{E}X)^2$ and deduce the following improvement of the estimate in a):

$$\mathsf{P}(X = 0) \leq \frac{\mathsf{Var}(X)}{\mathsf{E}(X^2)}.$$

[**Hint:** Apply the result in b) to $X \equiv XY$ with $Y = \mathbb{1}_{\{X>0\}}$.]

Q4 A rooted tree R_d of index $d > 2$ is a tree, in which every vertex different from the root r has exactly d neighbours; e.g.,



shows a finite part of R_3.

a) Carefully define the site percolation model on R_d.

b) Prove that $\theta_{\mathsf{r}}^{\mathsf{site}}(p) \equiv \mathsf{P}_p\big(\mathsf{r} \overset{\mathsf{site}}{\leftrightsquigarrow} \infty\big) > 0$ if and only if $(d-1)p > 1$.

[**Hint:** Find a recurrence for $\rho_n := \mathsf{P}(\{\mathsf{r} \overset{\mathsf{site}}{\leftrightsquigarrow} \text{ level } n\}^{\mathsf{c}} \mid \mathsf{r} \text{ is open})$.]

c) Is it true that $\theta_{\mathsf{r}}^{\mathsf{site}}(p) = 0$ if and only if $\theta_{\mathsf{v}}^{\mathsf{site}}(p) = 0$ for each vertex v of R_d? If so, what is the value of the critical site percolation probability on R_d?

SECTION B

Q5 An *n-step path* s_0, s_1, …, s_n of integers satisfies $|s_{k+1} - s_k| = 1$ for $0 \le k \le n-1$. For x, $y \in \mathbb{Z}$, let $N_n(x, y)$ denote the number of n-step paths with $s_0 = x$ and $s_n = y$.

a) Find a closed formula for $N_n(x, y)$.

b) For positive integers a and b, show that the number of n-step paths with $s_0 = a$ and $s_n = b$ that visit 0 equals $N_n(-a, b)$.

c) For positive integers a and b, show that the number of n-step paths such that

$$s_0 = 0, \quad s_1 > -a, \quad s_2 > -a, \dots, \quad s_{n-1} > -a, \quad s_n = b$$

equals $N_n(0, b) - N_n(0, 2a + b)$.

d) If $a > b > 0$ are integers, show that the number of n-step paths such that $s_0 = 0$, $s_1 < a$, $s_2 < a$, …, $s_{n-1} < a$, $s_n = b$ equals $N_n(0, b) - N_n(0, 2a - b)$.

Q6 Let $(X_n)_{n \ge 1}$ be independent identically distributed random variables with $\mathsf{E}(X_k) = \mu$ and $\mathsf{Var}(X_k) = \sigma^2 < \infty$. Denote $S_n = X_1 + \cdots + X_n$.

a) Use Chebyshev's inequality and the sufficient condition of almost sure convergence to show that

$$\frac{1}{m^2} S_{m^2} \equiv \frac{1}{m^2}\big(X_1 + \cdots + X_{m^2}\big) \xrightarrow{\text{a.s.}} \mu = \mathsf{E}(X_k) \qquad \text{as } m \to \infty.$$

b) Assuming $X_k \ge 0$, show that $(m+1)^{-2} S_{m^2} \le n^{-1} S_n \le m^{-2} S_{(m+1)^2}$ provided $m^2 \le n \le (m+1)^2$. Use this inequality to show that $\frac{1}{n} S_n \to \mu = \mathsf{E}(X_k)$ almost surely as $n \to \infty$ for non-negative random variables $X_k \ge 0$.

c) In the general case, decompose into positive and negative part, $X_k = X_k^+ - X_k^-$, where $X_k^+ \ge 0$ and $X_k^- \ge 0$ and deduce the *strong law of large numbers*:

if $S_n = X_1 + \cdots + X_n$ where $(X_k)_{k \ge 1}$ are i.i.d. random variables satisfying $\mathsf{E}\big((X_k)^2\big) < \infty$, then $n^{-1} S_n \to \mu = \mathsf{E}(X_k)$ almost surely as $n \to \infty$.

Q7 An isolated edge in a graph G is a pair of vertices u, v in G that are adjacent to each other, but to no other vertices in G.

a) Calculate the expected number of isolated edges in a binomial random graph $G_{n,p}$.

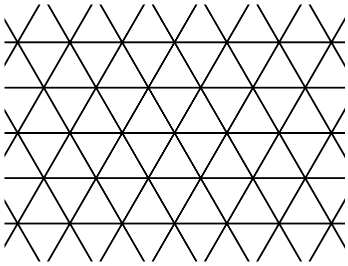
b) Calculate the variance of the number of isolated edges in a binomial random graph $G_{n,p}$.

c) Suppose $p = p(n)$ is a function satisfying $np/\log(n) \to c$ as $n \to \infty$, for some constant $c > 0$.

i) Prove that $\mathsf{P}(G_{n,p}$ has an isolated edge$) \to 0$ as $n \to \infty$ for $c > 1/2$.

ii) Prove that $\mathsf{P}(G_{n,p}$ has an isolated edge$) \to 1$ as $n \to \infty$ for $0 < c < 1/2$.

Q8 Consider bond percolation on the triangular lattice (see the picture below), with every bond independently open with probability $p \in [0, 1]$.



a) Carefully define the percolation probability $\theta(p)$; show that it is a non-decreasing function of p and hence define the critical value p_{c}.

b) Show that $\theta(p) = 0$ for $p > 0$ small enough; hence deduce that $p_{\mathsf{c}} \geq p'$ for some $p' > 0$.

c) Show that $\theta(p) > 0$ for $1 - p > 0$ small enough; hence deduce that $p_{\mathsf{c}} \leq p''$ for some $p'' < 1$.