Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH1051-WE01

Title:
Analysis I

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

1. (a) Let $a_1 = -2$, $a_2 = 1$, and for an integer $n \geq 2$ let

$$a_{n+1} = \frac{1}{2}(a_n + a_{n-1}).$$

Show that

$$a_n = 4 \cdot \left(-\frac{1}{2}\right)^n.$$

[6]

(b) Show that

$$\prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^k = \frac{n^n}{n!}$$

for every integer $n \geq 2$. **[4]**

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2. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be bounded sequences of positive numbers, and let

$$A = \{a_n \in \mathbb{R} \mid n \in \mathbb{N}\}, \qquad B = \{b_n \in \mathbb{R} \mid n \in \mathbb{N}\}.$$

Consider

$$C = \{a_n \cdot b_n \in \mathbb{R} \mid n \in \mathbb{N}\}.$$

(a) Show that $\sup(C) \leq \sup(A) \cdot \sup(B)$. **[4]**

(b) Give an example of bounded sequences of positive numbers $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ such that $\sup(C) < \sup(A) \cdot \sup(B)$. Justify your statement. **[6]**

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3. Calculate the limit, or show that the limit does not exist, of the following sequences. State any results that you are using.

(a) $x_n = \dfrac{(n+2)^2 - (n+1)^2}{n}$, **[3]**

(b) $y_n = \sqrt[n]{n^2+1}$, **[4]**

(c) $z_n = \left(1 - \dfrac{1}{2}\right) \cdot \left(1 - \dfrac{1}{3}\right) \cdots \left(1 - \dfrac{1}{n}\right)$. **[3]**

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4. (a) Determine lim inf and lim sup of the following sequence. Justify your statements.

$$x_n = \frac{n + (-1)^n(2n-1)}{2n + (-1)^{n+1}(n-1)}.$$

[4]

(b) Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be bounded sequences. Show that

$$\liminf_{n\to\infty}(x_n + y_n) \leq \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \leq \limsup_{n\to\infty}(x_n + y_n).$$

[6]

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5. (a) Decide whether the following series are convergent. State any results that you use.

(i) $\displaystyle\sum_{n=1}^{\infty} \binom{2n}{n} 2^{-n}$, **[3]**

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[4]{n^3}}$, **[3]**

(b) Show that the following equality holds.

$$\sum_{k=2}^{\infty} \log\left(1-\frac{1}{k^2}\right) = \log\left(\frac{1}{2}\right).$$ **[4]**

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6. (a) Give the ϵ-δ definition for a function f on an open set $X \subseteq \mathbb{R}$ to be continuous at $c \in X$. Also give the characterization of continuity in terms of limits. **[3]**

(b) Let $f(x)$ be the function on $(-1,1)$ defined by

$$f(x) = \begin{cases} x^2+2 & \text{if } x \le 0; \\ x^2+3 & \text{if } x > 0. \end{cases}$$

Show that f is *not* continuous at $x=0$ in two different ways, independently employing the two characterizations of continuity in (a). **[7]**

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7. (a) Carefully state the mean value theorem. **[3]**

(b) Explain why the mean value theorem applies to $f(x) = \sqrt{x}$ on the interval $[a,b]$ for any $0 \le a < b$. Explicitly verify the theorem. **[7]**

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8. (a) Show $\sum_{n=0}^{\infty} x^{3n+2} = \frac{x^2}{1-x^3}$ for $|x| < 1$. Use this identity to express

$$f(x) := \sum_{n=0}^{\infty} (3n+2)x^{3n+2}$$

explicitly as a rational function. Carefully justify your reasoning. **[7]**

(b) Denote the above function by $f(x)$. For which k does the k-th derivative $f^{(k)}(0)$ at $x=0$ vanish? **[3]**

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9. (a) Let $q_n = \sqrt[n]{2}$ and consider the non-equidistant partition of $[1,2]$ by $1 = q_n^0 < q_n < q_n^2 < \cdots < q_n^n = 2$, that is, $x_k = q_n^k, k = 0, \ldots, n$.
Define (with proof!) a sequence of step functions $f_n(x)$ associated to the above partitions which converges uniformly to $f(x) = \frac{1}{x}$. **[5]**

(b) Use these step functions $f_n(x)$ to compute $\int_1^2 \frac{1}{x} dx$ directly from the definition. (l'Hôpital will help.) **[5]**

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10. (a) Let $t \geq 0$. Use the growth theorem to show that

$$\log(1+t) \leq t.$$

Conclude that for all $x \in [a,b]$, a compact subset of $(0,\infty)$, and $k > 0$ we have

$$\log(1+e^{-kx}) \leq e^{-ka}.$$

[5]

(b) Show that

$$f(x) = \sum_{k=0}^{\infty} \log(1+e^{-kx})$$

defines a continuous function on $(0,\infty)$. Explain your reasoning. **[5]**
