Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH1061-WE01

Title:
Calculus I

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

1. (a) Find the general solution $y(x)$ to the ordinary differential equation

$$y'' + 4y' + 4y = \cos(2x).$$ **[6]**

(b) Find the general solution $y(x)$ to the ordinary differential equation

$$y'' + 4y' + 4y = e^{-2x}.$$ **[4]**

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2. Let

$$f(x) = \int_0^{x^2} \frac{1-t}{2+\cos(t)}\, dt,$$

(a) Find all the critical points of $f(x)$. For each critical point, using the first derivative test or otherwise, determine if it is a local minimum, local maximum, or a point of inflection. **[7]**

(b) Use Rolle's theorem to show that there exist at least 2 points in the interval $(-1, 1)$ such that $f''(x) = 0$. **[3]**

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3. Let D be the finite compact region between the curves $y = x^3 - 3x$ and $y = x$. Calculate

$$\iint_D (x+1)\, dx\, dy.$$ **[10]**

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4. (a) Show that $I = e^x$ is an integrating factor for the ODE

$$\left(y\left[x^2 + 2x + 1\right] + 2\right) dx + (x^2 + 1)\, dy = 0.$$ **[3]**

(b) Hence, or otherwise, solve the initial value problem

$$\left(y\left[x^2 + 2x + 1\right] + 2\right) dx + (x^2 + 1)\, dy = 0, \quad \text{with } y(0) = 1.$$

You may leave your answer in implicit form. **[7]**

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5. (a) Let $f(x) = 1 + x$ for $0 < x < \pi$. State the even extension of $f(x)$ to the interval $(-\pi, \pi)$, and hence calculate the half-range cosine series of $f(x)$. **[6]**

(b) By also considering the half-range sine series of $f(x)$, show that for all $x \in (0, \pi)$,

$$\sum_{n=1}^{\infty} \left(\frac{2\cos((2n-1)x)}{(2n-1)^2} + \frac{(1 + (1+\pi)(-1)^{n+1})\sin(nx)}{n} \right) = \frac{\pi}{2} + \frac{\pi^2}{4}.$$ **[4]**

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6. (a) A function $f(x, y)$ is such that its partial derivatives are given as

$$f_x = -y + \cos(x + 2y),$$
$$f_y = -x + 2\cos(x + 2y).$$

Use the Chain Rule to calculate $\frac{df}{dt}$ along the curve $(x, y) = (t, t^2)$, expressing your answer as a function of t. **[2]**

(b) Find and classify the stationary points of the function

$$f(x, y) = x^2y^2 - 3x^2y + 2x^2 - y^3 + 12y.$$ **[8]**

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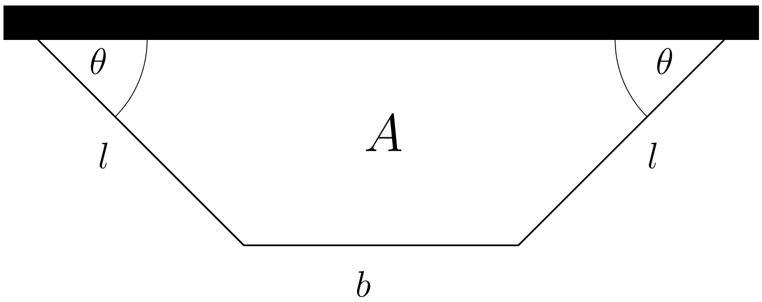
7. A gardener buys 9 metres of fencing in order to fence off a vegetable patch. One side of the patch lies against a wall, indicated by a thick black line in the figure below. The fencing is cut into three pieces, two of length l and a third of length b, which are to be arranged symmetrically as shown in the figure, so that the two pieces of length l make an angle θ with the wall. The gardener wishes to maximise the area A enclosed by the fence and wall.



(a) Find expressions for the enclosed area A, and the constraint on the total length of the fence (excluding the wall) in terms of l, b and θ. **[3]**

(b) Use the method of Lagrange multipliers to determine the choice of l, b and θ which maximises the area A enclosed using the three pieces of fence, and thus find the maximum value of A. **[7]**

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8. A differential operator $\mathcal{L}$ is defined as

$$\mathcal{L} = \frac{d^2}{dx^2} - 2x\frac{d}{dx}.$$

(a) Show that if y is twice differentiable then

$$\mathcal{L}y = e^{x^2}\frac{d}{dx}\left(e^{-x^2}\frac{dy}{dx}\right).$$ **[2]**

(b) Show that $\mathcal{L}$ is a self-adjoint operator with respect to the inner product

$$(f\,,\,g) = \int_{-\infty}^{\infty} e^{-x^2} f(x)g(x)dx,$$

where you may assume that f and g are twice differentiable and that they and their first derivatives go to zero faster than $\exp(-x^2/2)$ as $|x| \to \infty$. **[3]**

(c) By substituting the function $P_2(x) = x^2 + c$ into the equation $\mathcal{L}P_2 = \lambda P_2$, find the values of the constants c and λ such that $P_2(x)$ is an eigenfunction of $\mathcal{L}$. **[3]**

(d) Explain briefly why you would expect the integral $\int_{-\infty}^{\infty} P_2(x)\exp(-x^2)dx$ to be equal to zero. Use this to calculate $\int_{-\infty}^{\infty} x^2\exp(-x^2)dx$, given that $\int_{-\infty}^{\infty}\exp(-x^2)dx = \sqrt{\pi}$. **[2]**

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9. A bar of metal is arranged along the x-axis with its ends at $x = 0$ and $x = \pi$. The temperature of the bar $u(x,t)$ obeys the heat equation $u_t = k^2 u_{xx}$, and the ends are held at zero degrees so that $u(0,t) = u(\pi,t) = 0$.

(a) Show that the function

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx)\exp(-k^2n^2t)$$

solves the heat equation and satisfies the boundary conditions at the endpoints. **[2]**

(b) Initially the bar's temperature is given by

$$u(x,0) = \begin{cases} x, & 0 \le x < \pi/2, \\ \pi - x, & \pi/2 \le x \le \pi. \end{cases}$$

Find the solution for the temperature of the bar at later times $u(x,t)$ in the form of the series in part a) by determining the constants A_n in this case. **[4]**

(c) If instead the same total heat is distributed uniformly in the bar so initially its temperature is given by

$$u(x,0) = \pi/4$$

find the constants A_n in this case. **[2]**

(d) By using an approximation for the temperature distribution at late times in both the above cases, show that the ratio $U_1(t)/U_2(t) \to 4/\pi$ as $t \to \infty$, where $U_1(t)$, $U_2(t)$ are the temperatures at the midpoint of the bar in the cases described in part b) and c) respectively. **[2]**

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10. (a) State the Shift, Scaling and Derivative Theorems. [**3**]

(b) Find the Fourier transform of $\exp(-x^2/a)$ where a is a positive constant. [**2**]

(c) Find the Fourier transform of $\exp(-(x+c)^2)$ where c is a constant. [**1**]

(d) Find the Fourier transform of $(x+c)\exp(-(x+c)^2)$ where c is a constant. [**2**]

(e) Find the Fourier transform of $x\exp(-x^2-dx)$ where d is a constant. [**2**]

In parts b)-e) you may use that the Fourier transform of the Gaussian function $f(x)=\exp(-x^2)$ is

$$\tilde{f}(p)=\sqrt{\pi}\exp\left(-\frac{p^2}{4}\right).$$
