



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH1071-WE01
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Title: Linear Algebra I

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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1. In this question full marks cannot be obtained without showing all your working.

(a) Find the line of intersection of the two planes in \mathbb{R}^3 given by

$$x + 2y = 4 \quad \text{and} \quad x - y - z = 1.$$

Express your answer in both Cartesian and parameterised forms. [5]

(b) Find all unit vectors \mathbf{v} in \mathbb{R}^4 which are at an angle of $2\pi/3$ to the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad [5]$$

2. For what values of s does the following system of equations have a unique solution?

$$\begin{aligned} x + y - sz &= 1 \\ sx - sy + z &= 1 \\ x + (s + 1)y &= 1 \end{aligned}$$

Justify your answer, quoting any results from lectures that you use. [5]

For those values of s for which you think the system of equations does **not** have a unique solution, determine whether they have no solutions or infinitely many solutions. When there are infinitely many solutions, find the solution set. [5]

3. (a) Define what is meant by an *inverse* of a square matrix, and prove that a square matrix can have at most one inverse. [4]

(b) Define what is meant by a *vector subspace* of a vector space V . [2]

(c) Let

$$S = \left\{ p(x) \in \mathbb{R}[x]_2 : p'(1) = 0 \text{ and } \int_0^1 p(x) dx = 0 \right\},$$

where $p'(1)$ denotes the value of the derivative of $p(x)$ at 1. You may assume that S is a vector subspace of $\mathbb{R}[x]_2$, the space of polynomials of degree at most 2. Find a basis for S and give a basis for $\mathbb{R}[x]_2$ that includes your basis for S , justifying your answer. [4]

4. (a) Define what it means for a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_m\} \subset \mathbb{R}^n$ to be *linearly independent*. Define also the *span* of the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$. [2]
- (b) Prove that if $\{\mathbf{v}_1, \dots, \mathbf{v}_m\} \subset \mathbb{R}^n$ are linearly independent and $\mathbf{u} \in \mathbb{R}^n$ is not an element of their span, then $\{\mathbf{u}, \mathbf{v}_1, \dots, \mathbf{v}_m\}$ are also linearly independent. [4]
- (c) Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ are n vectors in \mathbb{R}^n . Let A be the $n \times n$ matrix whose columns are given by the \mathbf{v}_i . State and prove an 'if and only if' relation between the RREF of A and whether or not the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ are a basis for \mathbb{R}^n . You may assume that the set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ being a basis of \mathbb{R}^n is equivalent to them being linearly independent. [4]
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5. (a) Let V and W be vector spaces. Define what it means to say that a function $T: V \rightarrow W$ is a *linear map*, and define what is meant by the *image*, the *kernel*, the *rank* and the *nullity* of T . [5]
- (b) You are told that the function $T: \mathbb{R}[x]_2 \rightarrow \mathbb{R}[x]_3$ given by

$$T(p(x)) = p'(x) + 2p(x) + \int p(x)dx$$

is a linear map. Write down its matrix with respect to the standard bases of $\mathbb{R}[x]_2$ and $\mathbb{R}[x]_3$, and compute the rank and nullity of T . [5]

6. Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 1 \\ -1 & 2 & -1 \end{pmatrix}$$

and use it to find A^n for any integer $n \geq 3$. *Hint: Use the Cayley-Hamilton theorem.* [10]

7. Show that the matrix

$$A = \begin{pmatrix} -2 & 3 \\ -6 & 7 \end{pmatrix}$$

is similar to a diagonal matrix D . Find an appropriate matrix M that can convert A to D . [10]

8. Let $V = \mathbb{C}^2$. For $\mathbf{z} = (z_1, z_2)$ and $\mathbf{w} = (w_1, w_2)$ in V , define

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \bar{w}_1 + a z_2 \bar{w}_2 + b z_1 \bar{w}_2 + ic z_2 \bar{w}_1,$$

where in general $a, b, c \in \mathbb{C}$. Determine all values of a, b, c for which $\langle \mathbf{z}, \mathbf{w} \rangle$ is a complex inner product on V .

For such values of a, b, c , find all vectors $\mathbf{u} \in \mathbb{C}^2$ orthogonal to the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad [10]$$

9. Let $V = \mathbb{R}[t]_2$ denote the real vector space of all real polynomials in t of degree at most 2. Equip V with the inner product

$$\langle f, g \rangle = \int_0^1 2 f(t) g(t) dt, \quad (f, g \in V).$$

Let $U = \{f \in V : f(0) = 0 \text{ and } f(1) = 0\}$. Find a basis for the orthogonal complement U^\perp . [10]

10. Let

$$G = \{f_{a,b} : \mathbb{R} \rightarrow \mathbb{R} \mid f_{a,b}(x) = ax + b, a, b \in \mathbb{R}, a \neq 0\}.$$

Define a binary operation \bullet on G by *function composition*:

$$(f_{a,b} \bullet f_{c,d})(x) = f_{a,b}(f_{c,d}(x)).$$

Prove that (G, \bullet) is a group. Determine whether this group is abelian or non-abelian. [10]
